Things to know for the second midterm

The second exam will cover all of the material through October 22. Most of the subject matter will come from Chapter 3 of the notes, *Logic and sets*, and Chapter 4, *Proof techniques*.

In addition to the material from the first exam, you should be able to:

- translate between English sentences and logical expressions, especially those containing quantifiers;
- understand and use the quantifiers $\forall$ and $\exists$;
- understand and use bounded quantifiers such as $\forall x \in S (P(x))$;
- identify free and bound variables in a logical expression containing quantifiers;
- understand the meaning of logical statements with nested quantifiers;
- identify how the meaning of a statement with multiple quantifiers changes when the order of quantifiers changes;
- determine whether a statement involving quantifiers is true or false;
- find the negation of a logical expression, especially those containing one or more quantifiers;
- follow the logical structure of a proof;
- construct a direct proof;
- construct a proof by contrapositive;
- construct a proof by breaking into cases;
- construct a proof by contradiction;
- construct a proof of a statement with a universal quantifier;
- construct a proof of a statement with an existential quantifier.
Sample questions:

1. Write the negation of each of the following statements. Your answers should not contain the symbol \( \neg \).
   (a) \( \forall x (x > 2 \rightarrow x^2 > 4) \).
   (b) \( \exists x (x > 2 \leftrightarrow x^2 > 4) \).

2. Determine whether the following statements are True or False, and justify your answer. Here, the universe is the set of real numbers \( \mathbb{R} \).
   (a) \( \exists y \forall x (x^2 > y) \).
   (b) \( \forall x \exists y (x^3 > y) \).
   (c) \( \exists x \forall y (x^4 > y) \).
   (d) \( \forall y \exists x (x^5 > y) \).

3. Write the following English sentences as logical statements. You may assume that the universe is the set of integers \( \mathbb{Z} \).
   (a) There is an integer with no integer square root.
   (b) Every integer lies between two consecutive square numbers (inclusive).
   (c) An integer can only be divisible by 3 if its square is.
   (d) If the product of two integers is negative, then one of them must be positive and the other negative.

4. Write the negation of the following English sentences (as English sentences).
   (a) The square of any real number is positive.
   (b) There is an integer that is a divisor of every integer.
   (c) If two real numbers differ by at most 1, then their squares differ by at most 100.
   (d) Every real number with a square root has a nonnegative square root.

5. For each of the following proofs, determine the statement being proved.
   (a) Suppose \( n \) is even, so that \( n = 2m \) for some integer \( m \). Then
      \[ n^2 + n = 4m^2 + 2m = 2(2m^2 + m). \]
      Suppose instead that \( n \) is odd, so that \( n = 2k + 1 \) for some integer \( k \). Then
      \[ n^2 + n = (4k^2 + 4k + 1) + (2k + 1) = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1). \]
      Since \( n \) is either even or odd, it follows that ...
   (b) If \( x = 1 \), then suppose \( y \) satisfies \( 1 + y = 1 \cdot y \). Subtracting \( y \) from both sides then gives \( 1 = 0 \), which is a contradiction.
      If \( x \neq 1 \), then let \( y = \frac{x}{x-1} \). Then
      \[ x + \frac{x}{x-1} = \frac{x^2 - x}{x-1} + \frac{x}{x-1} = \frac{x^2}{x-1} = x \cdot \frac{x}{x-1}. \]
      Therefore ...

6. Prove or disprove the following statements.
   (a) An integer is even if and only if its square is even.
   (b) An integer \( n \) is a multiple of both 6 and 10 if and only if it is a multiple of 60.
   (c) The product of any two irrational numbers is irrational. (If needed, you may assume that \( \pi \) is irrational.)
1. (a) \( \exists x (x > 2 \land x^2 \leq 4) \).
   (b) \( \forall x (x > 2 \iff x^2 \leq 4) \).

2. (a) This is true. For instance, let \( y = -1 \). Since the square of any number is nonnegative, \( x^2 \geq 0 > -1 \) for all \( x \).
   (b) This is true. For an arbitrary real number \( x \), take \( y = x^3 - 1 \). Then \( x^3 > x^3 - 1 = y \).
   (c) This is false. For an arbitrary real number \( x \), take \( y = x^4 \). Then \( x^4 \neq x^4 = y \).
   (d) This is true. For an arbitrary real number \( y \), let \( x = \sqrt[3]{y + 1} \). Then \( x^5 = y + 1 > y \).

3. (a) \( \exists x \forall y (x \neq y^2) \).
   (b) \( \forall x \exists y (y^2 \leq x \leq (y + 1)^2) \).
   (c) \( \forall x (\exists y (x = 3y) \rightarrow \exists z (x^2 = 3z)) \).
   (d) \( \forall x \forall y (xy < 0 \rightarrow ((x > 0 \land y < 0) \lor (x < 0 \land y > 0))) \).

4. (a) There exists a real number whose square is not positive.
   (b) For every integer, there exists some integer of which it is not a divisor.
   (c) There exist two real numbers that differ by at most 1 whose squares differ by more than 100.
   (d) There exists a real number with at least one square root such that all of its square roots are negative.

5. (a) For any integer \( n \), \( n^2 + n \) is even.
   (b) For any real number \( x \), there exists a real number \( y \) such that \( x + y = xy \) if and only if \( x \neq 1 \).

6. (a) This is true. If \( n \) is even, then \( n = 2m \) for some integer \( m \), so \( n^2 = 4m^2 = 2(2m^2) \) is even. If \( n \) is odd, then \( n = 2k + 1 \) for some integer \( k \), so \( n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 \) is odd.
   (b) This is false. For example, \( n = 30 = 6 \cdot 5 = 10 \cdot 3 \) is a multiple of both 6 and 10 but it is not a multiple of 60 (since \( 30 = 60 \cdot \frac{1}{2} \)).
   (c) This is false. We claim that \( \frac{1}{\pi} \) is irrational. Indeed, suppose for the sake of contradiction that \( \frac{1}{\pi} = \frac{m}{n} \) for integers \( m \) and \( n \neq 0 \). Then also \( m \neq 0 \) (since \( \frac{1}{\pi} \neq 0 \)), so we can then write \( \pi = \frac{n}{m} \). But this would imply that \( \pi \) is rational, which we are assuming is false. It follows that \( \frac{1}{\pi} \) is irrational. Then the product of the two irrational numbers \( \pi \) and \( \frac{1}{\pi} \) is 1, which is rational.