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1. (10 points) Match each of the logical statements below with the correct truth value and explanation. (Each blank should be filled with one of the letters A–E.)
Assume the universe is the set $U = \{ x \in \mathbb{R} \mid 0 < x \leq 1 \}$.

\[
\begin{align*}
\text{A} & : \quad \forall x \exists y (x < y) & & \text{True: let } y = 1. \\
\text{B} & : \quad \exists x \forall y (x < y) & & \text{False: let } x = 1. \\
\text{C} & : \quad \forall y \exists x (x < y) & & \text{True: let } x = \frac{1}{2}, y = 1. \\
\text{D} & : \quad \exists y \forall x (x \leq y) & & \text{True: let } x = \frac{y}{2}. \\
\text{E} & : \quad \exists x \exists y (x < y) & & \text{False: let } y = \frac{x}{2}.
\end{align*}
\]

2. Write the negation of each of the following English sentences (as an English sentence). Your answers should not contain the word “not” or any similar negating words.

(a) (3 points) No even number is prime.

Solution: There is an even prime number.

(b) (3 points) If the sum of two real numbers is rational, then their product is rational.

Solution: There exist two real numbers whose sum is rational but whose product is irrational.

(c) (4 points) There is a rational number greater than 1 that when raised to any rational power results in a rational number.

Solution: Every rational number greater than 1 can be raised to some rational power to result in an irrational number.
3. Assume that $U = \mathbb{Z}$, and let $n$ be an integer. Consider the following definition:

“An integer $k$ is a quadratic residue modulo $n$ if $a^2 - k$ is a multiple of $n$ for some integer $a$.”

(a) (3 points) Write a logical formula that is equivalent to: “$k$ is a quadratic residue modulo $n$.”
(Your answer should not contain division or the “divides” symbol.)

Solution: $\exists a \exists b (a^2 - k = bn)$.

(b) (3 points) Write a logical formula that is equivalent to: “$k$ is not a quadratic residue modulo $n$.”
(Your answer should not use the symbol $\neg$, though you may use the symbol $\neq$.)

Solution: $\forall a \forall b (a^2 - k \neq bn)$.

(c) (4 points) Is 3 a quadratic residue modulo 11? Explain.

Solution: Yes. For instance, $5^2 - 3 = 22 = 2 \cdot 11$ is a multiple of 11.
4. For each of the following proofs, determine the statement being proved.

(a) (3 points) We prove the contrapositive. Suppose \( a = 2k + 1 \) and \( b = 2\ell + 1 \) for some integers \( k \) and \( \ell \). Then

\[
ab = (2k + 1)(2\ell + 1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell) + 1.
\]

**Solution:** If the product of two integers is even, then (at least) one of them must be even.

(b) (3 points) Since \( n \) is a square number, we can write \( n = k^2 \) for some integer \( k \). If \( k \) is odd, say \( k = 2a + 1 \), then

\[
n = (2a + 1)^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1
\]

is odd, which is a contradiction. Thus \( k \) is even, say \( k = 2b \). Then letting \( c = b^2 \), we have \( n = k^2 = 4b^2 = 4c \).

**Solution:** If \( n \) is an even square number, then \( n \) is a multiple of 4.

(c) (4 points) Let \( N \) be any integer such that \( N > \frac{1}{2x} \). Then

\[
N(x + \epsilon) - N(x - \epsilon) = 2N\epsilon > 1,
\]

so there must exist an integer \( m \) between \( N(x - \epsilon) \) and \( N(x + \epsilon) \). Thus \( \frac{m}{N} \in \mathbb{Q} \) lies between \( x - \epsilon \) and \( x + \epsilon \), or equivalently, \( |x - \frac{m}{N}| < \epsilon \).

**Solution:** For any real number \( x \) and any \( \epsilon > 0 \), there exists a rational number within \( \epsilon \) of \( x \). (In other words, for any real number \( x \), there exists a rational number arbitrarily close to \( x \).)
5. Prove the following statements.

(a) (5 points) The cube root of an irrational number is irrational.

Solution: Let \( r \) be an irrational number, and suppose for the sake of contradiction that \( \sqrt[3]{r} \) is rational. Then we can write \( \sqrt[3]{r} = \frac{a}{b} \) for some integers \( a \) and \( b \neq 0 \). Then \( r = (\sqrt[3]{r})^3 = \frac{a^3}{b^3} \in \mathbb{Q} \). This is a contradiction since we assumed \( r \) is irrational, so it follows that \( \sqrt[3]{r} \) must be irrational.

(b) (5 points) If \( A \) and \( B \) are disjoint sets and \( C \subseteq A \cup B \), then \( C \setminus A = B \cap C \).

Solution: Suppose \( x \in C \setminus A \). Then \( x \in C \) and \( x \notin A \). Since \( C \subseteq A \cup B \), we must then have \( x \in A \cup B \), so \( x \in A \) or \( x \in B \). But \( x \notin A \), so we must have \( x \in B \). Hence \( x \) is in both \( B \) and \( C \), so \( x \in B \cap C \). It follows that \( C \setminus A \subseteq B \cap C \).

Conversely, suppose \( x \in B \cap C \), so \( x \in B \) and \( x \in C \). If also \( x \in A \), then \( x \) would lie in both \( A \) and \( B \), which is a contradiction since \( A \) and \( B \) are disjoint. Hence \( x \notin A \). Together with \( x \in C \), this implies \( x \in C \setminus A \). Thus \( B \cap C \subseteq C \setminus A \), so we have \( C \setminus A = B \cap C \).