1. Answer all questions in the spaces provided.
2. Remember to justify your answers where appropriate.
3. No calculators, notes, or other outside assistance is allowed.
4. Each question is worth 10 points. The maximum score is 50.

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1. (a) (4 points) Which of the following relations are functions? Circle the ones that are. Do not justify your answer.

\[
\{(1, 2), (2, 3), (2, 1)\} \quad \{ (1, 2), (2, 3), (3, 2) \} \\
\{(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+ | a = b^2 \} \quad \{(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+ | a^2 = b \}
\]

(b) Find the domain and range of the functions given by the following formulas. (Assume \( x \) is a real number.) Do not justify your answer.

i. (3 points) \( f(x) = 1 - e^x \).

ii. (3 points) \( f(x) = \sqrt{x^2 - 1} \).
2. (a) For each function $f : \mathbb{R} \rightarrow \mathbb{R}$ given below, determine whether it is one-to-one, onto, both, or neither. Do not justify your answer.
   i. (2 points) $f(x) = x^{2/3}$.
   ii. (2 points) $f(x) = \frac{1}{1 + e^{-x}}$.

(b) Let $S = \{ x \in \mathbb{R} \mid 0 \leq x \leq 1 \}$. Give an example of a function that satisfies the following properties. Do not justify your answer.
   i. (3 points) A function $f : S \rightarrow S$ that is one-to-one but not onto.
   ii. (3 points) A function $f : S \rightarrow S$ that is onto but not one-to-one.
3. Let $A$ and $B$ be sets. Give the definitions of the following concepts. (Each definition will rely on the previous ones.)

(a) (2 points) The Cartesian product of $A$ and $B$.

(b) (2 points) A relation from $A$ to $B$.

(c) (3 points) A function from $A$ to $B$.

(d) (3 points) A bijection from $A$ to $B$. (Please do not use the terms one-to-one, onto, injective, or surjective.)
4. (10 points) Let \( n \geq 2 \) be an integer. Conjecture a formula for

\[ 2 \cdot \left( 1 - \frac{1}{4} \right) \left( 1 - \frac{1}{9} \right) \left( 1 - \frac{1}{16} \right) \cdots \left( 1 - \frac{1}{n^2} \right) \]

and prove it.

Write your formula here: _________________________________
5. (10 points) Define a sequence by $a_0 = 1$, $a_1 = 2$, $a_2 = 5$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$.
Determine for which natural numbers $n$ the inequality $a_n < 2^n$ holds, and prove it.

Write your answer here: _______________________________