MA 225 – Final
Fall 2019

Time: 3 hrs.

1. Answer all questions in the spaces provided.
2. Remember to justify your answers where appropriate.
3. No calculators, notes, or other outside assistance is allowed.
4. Each question is worth 10 points. The maximum score is 100.

Question | Points | Score
--- | --- | ---
1 | 10 | 
2 | 10 | 
3 | 10 | 
4 | 10 | 
5 | 10 | 
6 | 10 | 
7 | 10 | 
8 | 10 | 
9 | 10 | 
10 | 10 | 
Total: | 100 | 

Name: ____________________________
1. (10 points) Determine whether the following statements are true or false. Do not justify your answers.

________ The statement \( P \) implies the converse of \( P \rightarrow Q \).

________ If \( S \) and \( T \) are disjoint, then there does not exist a set \( U \) that is a subset of both \( S \) and \( T \).

________ If \( f \), \( g \), and \( h \) are functions such that \( f \) is one-to-one and \( f \circ g = f \circ h \), then \( g = h \).

________ The relation \( \{(x, y) \in \mathbb{R}^+ \times \mathbb{R} \mid x^2 - y^2 = 0\} \) is a function.

________ If \( f \) is a bijection, then \( f^{-1} \) must also be a bijection.

2. For an integer \( n \), let \( P(n) \) be the statement:

“If \( n \) is odd and a square, then \( n - 1 \) is a multiple of 8.”

(a) (8 points) Place a check mark next to the statements that are equivalent to \( P(n) \).

________ If \( n - 1 \) is not a multiple of 8, then \( n \) is even or not a square.

________ The number \( n - 1 \) is a multiple of 8 only if \( n \) is odd and a square.

________ If \( n \) is a square and \( n - 1 \) is not a multiple of 8, then \( n \) is even.

________ Either \( n \) is even and not a square, or \( n - 1 \) is a multiple of 8 (or both).

(b) (2 points) Let

\[
A = \{n \mid n \text{ is odd}\}, \\
B = \{n \mid n \text{ is a square}\}, \\
C = \{n \mid n - 1 \text{ is a multiple of 8}\}.
\]

Write a mathematical expression involving \( A \), \( B \), and \( C \) that is equivalent to \( \forall n P(n) \).
(Your statement should not contain any variable names other than \( A \), \( B \), or \( C \).)
3. Let \( A = \{1, 3, 5\} \) and \( B = \{2, 3\} \).
   (a) (2 points) List the elements of \( A \setminus B \).

   (b) (2 points) List the elements of \( B \times A \).

   (c) (3 points) Write down an expression for the set \( \{1, 2, 5\} \) using only \( A, B \), and set operations.

   (d) (3 points) How many different functions \( f : B \rightarrow A \) are there?
4. An integer is called \textit{squarefree} if it is not an integer multiple of any square number greater than 1. (For this problem, assume that the universe is $\mathbb{Z}$.)

(a) (4 points) Write a logical formula $P(n)$ equivalent to the statement “$n$ is squarefree.”

(b) (3 points) List the elements of the set

\[ \{ n \mid P(n) \text{ and } 25 \leq n \leq 30 \}. \]

(c) (3 points) Consider the statement:

“For all integers $n$, if $n$ is squarefree, then $n$ is not a square number.”

Is this statement true? Explain.
5. (a) (5 points) Translate the statement
\[ \exists a \forall b \exists c (b = ac) \]
into a sensible English sentence, and determine whether it is true or false, remembering to justify your answer. (Assume the universe is \( \mathbb{N} \).)

(b) (5 points) Let \( f : A \to B \). Write a logical expression that means “\( f \) is not onto,” and give an equivalent English definition. (Do not use the symbol \( \neg \). The symbol \( \neq \) is okay.)
6. Let \( f(x) = \frac{1}{1+x^2} \). (Assume the universe is \( \mathbb{R} \).)

(a) (2 points) Find the domain \( A \) of \( f \).

(b) (3 points) Find a set \( B \) such that \( f: A \to B \) is onto.

(c) (3 points) Is \( f \) one-to-one? Explain.

(d) (2 points) Suppose \( g(x) = x - 1 \). Give formulas for \( f \circ g \) and \( g \circ f \).
7. For each of the following proofs, determine the statement being proved.

(a) (3 points) Suppose the hypothesis holds. If \(a\) and \(b\) were both odd, then their sum would be even, so we must have that either \(a\) or \(b\) is even. Therefore their product must also be even.

(b) (3 points) Suppose \(f(a) = \frac{a+1}{2} = 1\) for some \(a \neq 2\). Multiplying both sides by \(a - 2\) gives \(a + 1 = a - 2\), and then subtracting \(a\) gives \(1 = -2\), which is a contradiction.

Now choose any \(b \neq 1\), and let \(a = \frac{3}{b-1} + 2\) (which is defined and not equal to 2). Then

\[
f(a) = \frac{3}{b-1} + 3 = \frac{3 + 3(b-1)}{3} = \frac{3b}{3} = b.
\]

(c) (4 points) We proceed by strong induction on \(n\). The base case \(n = 0\) is trivial. Now choose an arbitrary \(n\), and assume the claim holds for all natural numbers less than \(n\). Let \(k\) be the largest integer less than or equal to \(\log_2 n\), so that \(k \leq \log_2 n < k + 1\). Then \(2^k \leq n < 2^{k+1}\), so \(0 \leq n - 2^k < 2^k\). By the inductive hypothesis, \(n - 2^k\) can be written as a sum of distinct powers of 2, none of which can equal \(2^k\) since \(n - 2^k < 2^k\). Then adding \(2^k\) to this sum gives the claim for \(n\).
8. (10 points) Define a sequence of numbers by \(a_1 = 1\), and \(a_n = \frac{a_{n-1}}{1 + a_{n-1}}\) for \(n \geq 2\).

Conjecture and prove a general formula for \(a_n\).
9. (10 points) Prove that at least one of $\pi + e$ and $\pi - e$ is irrational.

   You may assume that both $\pi$ and $e$ are irrational.
10. (10 points) Show that the function \( f : \mathbb{R} \leq 1 \rightarrow \mathbb{R} \) defined by \( f(x) = \ln(1 - x) \) is a bijection, and find its inverse.