MA 225 Homework 2 – Solutions

1. (a) This is false. For example, let \( a = b = 0 \) and \( c = d = -1 \), which satisfy \( a \geq c \) and \( b \geq d \). However,

\[
ab = 0 \not\geq 1 = cd.
\]

(b) This is true. Note that \( a \geq c \) implies \( a - c \geq 0 \). Therefore, we can multiply both sides of the inequality \( b \geq d \) by \( a - c \) without changing the direction of the inequality, which gives

\[
ab - bc = (a - c)b \geq (a - c)d = ad - cd.
\]

Adding \( bc + cd \) to both sides gives

\[
ab + cd \geq ad + bc,
\]

as desired.

2. Yes, such a sequence of numbers does exist. For instance, let \( a_1 = 100 \), \( a_2 = 2 \), and \( a_3 = a_4 = \cdots = a_{100} = 1 \). Then

\[
a_1 + a_2 + a_3 + \cdots + a_{100} = 100 + 2 + 98 \cdot 1 = 200,
\]

which equals

\[
a_1a_2a_3\cdots a_{100} = 100 \cdot 2 \cdot 1^{98} = 200.
\]

3. Fix real numbers \( a \) and \( b \). The second axiom gives us the following two equalities (by letting \( x = a \) and \( y = b \) for the first, and by letting \( x = b \) and \( y = a \) for the second):

\[
(a \ast b) \ast (a \ast b) = b \ast a,
\]

\[
(b \ast a) \ast (b \ast a) = a \ast b.
\]

Consider \( c = ((a \ast b) \ast (a \ast b)) \ast ((a \ast b) \ast (a \ast b)) \). We can use the equalities above to simplify:

\[
c = ((a \ast b) \ast (a \ast b)) \ast ((a \ast b) \ast (a \ast b))
\]

\[
= (b \ast a) \ast (b \ast a)
\]

\[
= a \ast b.
\]

But we could instead simplify \( c \) by letting \( x = y = a \ast b \) in the second axiom, which gives:

\[
c = ((a \ast b) \ast (a \ast b)) \ast ((a \ast b) \ast (a \ast b))
\]

\[
= (a \ast b) \ast (a \ast b)
\]

\[
= b \ast a.
\]

It follows that \( a \ast b = c = b \ast a \).