MA 225 Homework 6 – Solutions

1. (a) This statement can be written as $\forall x(x^2 > 4 \rightarrow x > 2)$. Recall that the negation of $P \rightarrow Q$ is $P \land \neg Q$. Therefore, the negation of the original statement is

$$\exists x(x^2 > 4 \land x \leq 2).$$

In words, “There exists a real number at most 2 whose square is greater than 4.”

(b) A largest real number $x$ would be one that is at least as large as every number $y$. In other words, the original statement is $\exists x \forall y(x \geq y)$. Its negation is then

$$\forall x \exists y(x < y),$$

or in words, “For every real number, there is a larger real number.”

(c) To disprove the statement, we would need to produce a real number that has a square root but that does not have (at least) two square roots; in other words, it must have exactly one square root. Therefore, the negated statement is

$$\exists x \exists !y(x = y^2),$$

or “There exists a real number with a unique square root.”

(d) To say that something can be arbitrarily close to 0 means that no matter how close we want to get (say, within some $a > 0$), one can find an instance that is less than $a$. Hence the original statement can be written $\forall a > 0 \exists x(x + \frac{1}{x} < a)$. Its negation is

$$\exists a > 0 \forall x(x + \frac{1}{x} \geq a),$$

or in words, “There is some positive number $a$ such that the sum of a positive real number and its reciprocal is always at least $a$.”

2. (a) Claim: Let $x$ and $y$ be positive real numbers. If $x + y \leq 2$, then $xy \leq 1$.

Proof. Suppose $x + y \leq 2$.

We can multiply both sides of the inequality above by the positive number $x$ to get $x^2 + xy \leq 2x$.

We can rewrite this inequality as

$$xy \leq 2x - x^2 = 1 - (x^2 - 2x + 1) = 1 - (x - 1)^2.$$

The square of any real number is nonnegative, so $(x - 1)^2 \geq 0$.

Thus

$$xy \leq 1 - (x - 1)^2 \leq 1.$$

Therefore, if $x + y \leq 2$, then $xy \leq 1$. \qed

(b) Claim: Let $x$ and $y$ be positive real numbers. If $x^2 - y^2 \leq 1$, then $x - y \leq 1$. 
Proof. Suppose $x - y > 1$. (We will prove the contrapositive statement.)

Since $x$ and $y$ are both positive, so is $x + y$.
Therefore we can multiply the inequality above by $x + y$ to get

$$x^2 - y^2 = (x + y)(x - y) > x + y.$$ 

Since we know $y$ is positive and $x - y > 1$, we find that

$$x^2 - y^2 > x + y = (x - y) + 2y > x - y > 1.$$ 

Therefore, if $x^2 - y^2 \leq 1$, then $x - y \leq 1$. \qed