MA 225 Homework 9 – Solutions

1. We proceed by strong induction on $n$. First, we verify that it is possible to make 18, 19, 20, and 21 cents of change:

\[
\begin{align*}
18 &= 4 + 7 + 7, \\
19 &= 4 + 4 + 4 + 7, \\
20 &= 4 + 4 + 4 + 4 + 4, \\
21 &= 7 + 7 + 7.
\end{align*}
\]

Now pick any integer $n \geq 22$, and suppose that it is possible to make $k$ cents of change for all $18 \leq k < n$. Then in particular, it is possible to make $n - 4$ cents of change, so we can make $n$ cents of change by adding a 4-cent coin. The result then follows by strong induction.

2. Let $g(n)$ be the number of ways to cover a $2 \times n$ rectangle with $n$ dominoes. We will use strong induction on $n$ to prove that $g(n) = F_{n+1}$.

For the base cases $n = 1$ and $n = 2$, we note that there is $F_2 = 1$ way to cover a $2 \times 1$ rectangle (the whole rectangle is a domino), and $F_3 = 2$ ways to cover a $2 \times 2$ rectangle (by splitting vertically or horizontally), as shown below.

```
  
  
```

Now let $n \geq 3$ be any integer. In any covering of the $2 \times n$ rectangle, the upper left corner must either be covered by a vertical domino or a horizontal domino.

- If it is covered by a vertical domino, then there remains a $2 \times (n - 1)$ rectangle to be covered. Thus there are $g(n - 1)$ ways to complete the covering in this case.

```
  2 \times (n - 1)
```

- If it is covered by a horizontal domino, then the bottom left corner must also be covered by a horizontal domino, leaving a $2 \times (n - 2)$ rectangle to be covered. Thus there are $g(n - 2)$ ways to complete the covering in this case.

```
  2 \times (n - 2)
```

It follows that $g(n) = g(n - 1) + g(n - 2)$. By the inductive hypothesis, we then have

\[
g(n) = g(n - 1) + g(n - 2) = F_n + F_{n-1} = F_{n+1},
\]

as desired.
3. (a) Since $1 < \frac{m}{n} < 2$, multiplying by $n$ gives $n < m < 2n$. Then $n < m$ implies $m - n > 0$, and $m < 2n$ implies $2n - m > 0$ as well as $m - n < n$. Thus if we let $m' = 2n - m$ and $n' = m - n$, both $m'$ and $n'$ are positive and $n' < n$.

Now

$$s = \frac{2 - r}{r - 1} = \frac{2 - \frac{m}{n}}{\frac{m}{n} - 1} = \frac{2n - m}{m - n} = \frac{m'}{n'},$$

as desired.

(b) Suppose for the sake of contradiction that $\sqrt{2}$ is rational. Then we can write $\sqrt{2} = \frac{m}{n}$ for positive integers $m$ and $n$ with $n$ minimal (using the well-ordering principle). Since $1 < \sqrt{2} < 2$, letting $r = \sqrt{2}$ in part (a) means that we can write

$$s = \frac{2 - \sqrt{2}}{\sqrt{2} - 1} = \frac{\sqrt{2}(\sqrt{2} - 1)}{\sqrt{2} - 1} = \sqrt{2},$$

so we have written $\sqrt{2}$ as a rational number $\frac{m'}{n'}$ with a smaller denominator. This contradicts the minimality of $n$, so $\sqrt{2}$ cannot be rational.

4. We first show that $A \times (B \setminus C) \subseteq (A \times B) \setminus (A \times C)$. Choose an arbitrary element $(x, y) \in A \times (B \setminus C)$. Then $x \in A$ and $y \in B \setminus C$, so $y \in B$ and $y \notin C$. Since $x \in A$ and $y \in B$, $(x, y) \in A \times B$. We cannot have $(x, y) \in A \times C$ for this would imply $y \in C$, which is a contradiction (since we know $y \notin C$), so we must have $(x, y) \notin A \times C$. Thus $(x, y) \in (A \times B) \setminus A \times C$.

Now we show that $(A \times B) \setminus (A \times C) \subseteq A \times (B \setminus C)$. Choose an arbitrary element $(x, y) \in (A \times B) \setminus (A \times C)$. Then $(x, y) \in A \times B$ and $(x, y) \notin A \times C$. Since $(x, y) \in A \times B$, we have $x \in A$ and $y \in B$. We cannot have $y \in C$, for if we did, then with $x \in A$, this would imply $(x, y) \in A \times C$, which we know is not the case. Thus $y \notin C$. It follows that $y \in B \setminus C$, so $(x, y) \in A \times (B \setminus C)$. 
