MA 225 Worksheet 11 – 11/14

Are the following relations functions? If so, determine their domain and range.

1. \( \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \).
   This is not a function since it contains both \((0, 1)\) and \((0, -1)\).

2. \( \{(x, y) \in \mathbb{R}^2 \mid x = e^y\} \).
   This is a function. Its domain is \(\mathbb{R}^+\) and its range is \(\mathbb{R}\). (We usually write this function as \(f(x) = \ln x\).)

3. \( \{(x, y) \in \mathbb{R}^2 \mid xy + x + y = 0\} \).
   This is a function. Its domain and range are both \(\mathbb{R} \setminus \{-1\}\). (We usually write this function as \(f(x) = -\frac{x}{x+1} = -1 + \frac{1}{x+1}\).)

Determine whether the following functions are (a) one-to-one and/or (b) onto.

4. \(f : \mathbb{R} \rightarrow \mathbb{R}\), where \(f(x) = e^x\).
   This function is one-to-one (the only preimage of a positive number \(y\) is \(x = \ln y\)) but not onto (the range does not include any nonpositive numbers).

5. \(f : \mathbb{R} \rightarrow \mathbb{R}\), where \(f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{x} & \text{otherwise.} \end{cases}\)
   This function is both one-to-one and onto. The only preimage of \(y\) is, surprisingly, \(f(y)\), that is, \(\frac{1}{y}\) if \(y \neq 0\) and 0 if \(y = 0\).

6. \(f : \mathbb{Z} \rightarrow \mathbb{N}\), where \(f(x) = |x|\).
   This function is not one-to-one (for instance, \(f(1) = f(-1) = 1\), or more generally, \(f(a) = f(-a) = a\) for any positive integer \(a\)) but it is onto since every natural number is the absolute value of itself (or its negative).

Find bijections between the following pairs of sets.

7. \(\{x \in \mathbb{R} \mid 0 < x < 1\}\) and \(\{x \in \mathbb{R} \mid 4 < x < 6\}\).
   One possibility is \(f(x) = 4x + 2\).

8. The set of integers \(\mathbb{Z}\) and the set of even integers \(2\mathbb{Z}\).
   One possibility is \(f(x) = 2x\).

9. \(\{x \in \mathbb{R} \mid 0 < x < 1\}\) and \(\mathbb{R}^+\).
   One possibility is \(f(x) = \frac{1}{x} - 1\). (Another is \(f(x) = \tan\left(\frac{\pi}{2}\right)\).)