Let $A$ be a set, and let $R \subseteq A \times A$ be a relation on $A$. Here are the definitions of some properties that $R$ sometimes satisfies:

- **reflexivity**: for all $x \in A$, $(x, x) \in R$.
- **symmetry**: for all $x, y \in A$, if $(x, y) \in R$, then $(y, x) \in R$.
- **anti-symmetry**: for all $x, y \in A$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.
- **transitivity**: for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

We say that $R$ is an *equivalence relation* if it satisfies reflexivity, symmetry, and transitivity. We say that $R$ is a *partial order* if it satisfies reflexivity, anti-symmetry, and transitivity.

Suppose $A = \{1, 2, 3, \ldots, 100\}$. Which of the following relations are equivalence relations? Partial orders?

1. $\{(x, y) \in A \times A \mid x \leq y\}$.
2. $\{(x, y) \in A \times A \mid x < y\}$.
3. $\{(x, y) \in A \times A \mid y - x \text{ is even}\}$.
4. $\{(x, y) \in A \times A \mid y - x \text{ is odd}\}$.
5. $\{(x, y) \in A \times A \mid y \text{ is a multiple of } x\}$.
6. $\{(x, y) \in A \times A \mid y \text{ is a multiple of } x \text{ or } x \text{ is a multiple of } y\}$.
7. $\{(x, y) \in A \times A \mid \frac{y}{x} = 2^k \text{ for some integer } k\}$.

Which relations are both an equivalence relation and a partial order? Formulate a theorem and prove it.
1. This is a partial order. Indeed, for all $x$, $y$, and $z$,
   
   - $x \leq x$ for all $x$;
   - if $x \leq y$ and $y \leq x$, then $x = y$;
   - if $x \leq y$ and $y \leq z$, then $x \leq z$.

   It is not symmetric since $1 \leq 2$ but $2 \not\leq 1$.

2. This is neither (it is not reflexive).

3. This is an equivalence relation:
   
   - $x - x = 0$ is even for all $x$;
   - if $y - x$ is even, then $x - y = -(y - x)$ is even;
   - if $y - x$ is even and $z - y$ is even, then so is $z - x = (z - y) + (y - x)$.

   It is not anti-symmetric: $3 - 1$ and $1 - 3$ are even but $1 \neq 3$.

4. This is neither (it is not reflexive).

5. This is a partial order:
   
   - $x$ is a multiple of $x$ for all $x$;
   - if $y$ is a multiple of $x$ and $x$ is a multiple of $y$, then $x = y$;
   - if $y$ is a multiple of $x$ and $z$ is a multiple of $y$, then $z$ is a multiple of $x$.

   It is not symmetric since $2$ is a multiple of $1$ but not vice versa.

6. This is neither: it is not transitive since $2$ and $3$ are both multiples of $1$, but neither $2$ nor $3$ is a multiple of the other.

7. This is an equivalence relation:
   
   - $\frac{y}{x} = 2^0$ for all $x$;
   - if $\frac{y}{x} = 2^k$, then $\frac{z}{y} = 2^{-k}$;
   - if $\frac{y}{x} = 2^k$ and $\frac{z}{y} = 2^\ell$, then $\frac{z}{x} = \frac{z}{y} \cdot \frac{y}{x} = 2^\ell \cdot 2^k = 2^{k+\ell}$.

   It is not antisymmetric since $\frac{2}{1} = 2^1$ and $\frac{1}{2} = 2^{-1}$, but $1 \neq 2$.

   In fact, the only relations that are both equivalence relations and partial orders are the identity relations $i_A = \{ (x, x) \mid x \in A \}$. It is easy to check that $i_A$ satisfies all four given properties.

   Conversely, suppose $R$ is both an equivalence and partial order, and take an arbitrary $(x, y) \in R$. By symmetry, we must have that $(y, x) \in R$, but then by antisymmetry, we must have $x = y$. Thus $R$ can only contain ordered pairs of the form $(x, x)$ for $x \in A$. By reflexivity, it must contain all such ordered pairs. Thus we must have $R = i_A$. 

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