Let $P$ be the statement:

“If $x = 5$, then $f(x) = 17$.”

Let $Q$ be the converse statement:

“If $f(x) = 17$, then $x = 5$.”

(Assume that we are working in the universe of the real numbers.)

1. (a) Suppose $f(x) = x^2 - 8$. Is $P$ always true? Is $Q$ always true?

(b) Suppose $f(x) = 8 - x^2$. Is $P$ always true? Is $Q$ always true?

2. For each statement below, determine whether it is equivalent to $P$ or $Q$.

(a) $x = 5$ implies $f(x) = 17$.

(b) $x = 5$ if $f(x) = 17$.

(c) $x = 5$ only if $f(x) = 17$.

(d) Only if $x = 5$ does $f(x) = 17$.

(e) $x = 5$ is a necessary condition for $f(x) = 17$.

(f) $x = 5$ is a sufficient condition for $f(x) = 17$.

(g) If $x \neq 5$, then $f(x) \neq 17$.

(h) Either $x = 5$ or $f(x) \neq 17$ (or both).

(i) It is not possible that $x = 5$ and $f(x) \neq 17$.

(j) $x \neq 5$ unless (possibly) $f(x) = 17$.

Solution:

1. (a) We have that $P$ is always true since $f(5) = 5^2 - 8 = 17$. However, $Q$ is not always true since $f(-5) = 17$ as well.

(b) Now, $P$ is not always true since $f(5) \neq 17$. However, $Q$ is always true! Indeed, since $x^2 \geq 0$, so $8 - x^2 \leq 8$, meaning that $f(x)$ can never equal 17. Thus $Q$ is vacuously true.

2. The statements equivalent to $P$ are (a), (c), (f), (i), and (j). All other statements—(b), (d), (e), (g), and (h)—are equivalent to $Q$. 

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