1. A quartic polynomial \( p \) satisfies \( p(n) = 2^n \) for \( n = 0, 1, 2, 3, 4 \). What is \( p(5) \)?

2. A cubic polynomial \( f \) satisfies \( f(2) = 0, f(3) = 2, f(4) = 9, \) and \( f(5) = 23 \). Find a formula for \( f(n) \).

3. Find a closed formula for \( 1^2 + 2^2 + 3^2 + \cdots + n^2 \).

Solution:

1. Since \( p \) has degree 4 and the difference operator \( \Delta \) reduces degree by 1, we must have that \( \Delta^5 p(n) = 0 \). Hence the difference table must look as shown below, so \( p(5) = 31 \).

\[
\begin{array}{cccccc}
1 & 2 & 4 & 8 & 16 & 31 & \ldots \\
1 & 2 & 4 & 8 & 15 & \ldots \\
1 & 2 & 4 & 7 & \ldots \\
1 & 2 & 3 & \ldots \\
1 & 1 & \ldots \\
0 & \ldots \\
\end{array}
\]

2. Let \( g(n) = f(n + 2) \). Then \( g \) is also a cubic polynomial with difference table shown below.

\[
\begin{array}{cccccc}
0 & 2 & 9 & 23 & 46 & \ldots \\
2 & 7 & 14 & 23 & \ldots \\
5 & 7 & 9 & \ldots \\
2 & 2 & \ldots \\
0 & \ldots \\
\end{array}
\]

Since \( g \) is cubic, all lower rows must be 0. Hence \( g(n) = 2^{(n)} + 5^{(n)} + 2^{(n)} \). It follows that \( f(n) = g(n - 2) = 2^{(n-2)} + 5^{(n-2)} + 2^{(n-2)} \).

3. Let \( a_n = \sum_{k=1}^{n} k^2 \). Then \( a_{n+1} - a_n = (n + 1)^2 \), so the sequence \( \{a_n\} \) has difference table shown below.

\[
\begin{array}{cccccc}
0 & a_1 & a_2 & a_3 & a_4 & a_5 & \ldots \\
1 & 4 & 9 & 16 & 25 & \ldots \\
3 & 5 & 7 & 9 & \ldots \\
2 & 2 & \ldots \\
0 & 0 & \ldots \\
\end{array}
\]

Thus

\[
a_n = \binom{n}{1} + 3 \binom{n}{2} + 2 \binom{n}{3} = n + 3 \cdot \frac{n(n-1)}{2} + 2 \cdot \frac{n(n-1)(n-2)}{6} = \frac{n(n+1)(2n+1)}{6}.
\]