1. (a) Let $a_n$ be the number of ordered tuples $(a, b, c, d)$ of integers satisfying

$0 \leq a < b \leq c < d \leq n$.

Find a closed formula for $a_n$, as well as its ordinary generating function $A(x)$.

(b) Let $b_n$ be the number of ordered tuples $(A, B, C, D)$ of sets satisfying

$\emptyset \subseteq A \subsetneq B \subseteq C \subsetneq D \subseteq [n]$.

Find a closed formula for $b_n$, as well as its exponential generating function $B(x)$.

2. How many partitions $\lambda$ (of any size $|\lambda| \geq 0$) satisfy $\lambda_i \leq n - i$ for all $i \leq n$?

3. Two standard decks of 52 playing cards are independently shuffled and placed side by side. Which is more likely: that no card is in the same position in both decks, or that there is exactly one such card?

4. Give a combinatorial proof that

$$c(n+1, k+1) = \sum_{i=k}^{n} \frac{n!}{i!} \cdot c(i, k),$$

where $c(n, k)$ denotes the Stirling number of the first kind.

5. Let $a_n$ be the number of compositions of $n$ that do not contain a part of size exactly 2.

For example, $a_5 = 7$: $5, 41, 14, 311, 131, 113, 11111$.

Find a homogeneous linear recurrence with constant coefficients satisfied by $a_n$.

(You do not need to supply the initial conditions.)