### MA 524 – Midterm
**Fall 2015**
**Solutions**

Name: ____________________________________________________________

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1. Let $\lambda$ be a partition. A standard Young tableau of shape $\lambda$ is a filling of the boxes of $\lambda$ with the numbers $1, 2, \ldots, |\lambda|$ such that each row and column is strictly increasing. For instance,

$\begin{array}{ccc}
1 & 3 & 4 & 6 \\
2 & 5 \ \\
7
\end{array}$

is a standard Young tableau of shape $(4, 2, 1)$.

Find the number of standard Young tableaux of shape $\lambda$ if:

(a) (5 points) $\lambda = (m, 1, 1, \ldots, 1)$.

**Solution:** Clearly 1 must go in the upper left corner. Then the rest of the numbers in the first row can be any subset of $\{2, \ldots, m+n\}$ of size $m-1$ in increasing order, and the rest of the numbers in the first column will be the remaining numbers in increasing order. Hence the answer is $\binom{m+n-1}{m-1} = \binom{m+n-1}{n}$.

(b) (5 points) $\lambda = (n, n)$.

**Solution:** There is a bijection between standard Young tableaux of shape $(n, n)$ and Dyck paths of length $2n$: if $i$ is in the first row, take an up step, while if $i$ is in the second row, take a down step. Since each column is increasing, we can never have taken more down steps than up steps. It follows that the answer is the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$. 
2. Each of \( n \) shirts is to be dyed one of 7 colors: red, orange, yellow, green, blue, indigo, or violet. In how many ways can this be done if there must be at least one red shirt, at least one green shirt, and at least one blue shirt, and...

(a) (5 points) ...the shirts are indistinguishable?

**Solution:** Since the shirts are indistinguishable, we can dye any three shirts red, green, and blue. Then the remaining \( n - 3 \) shirts must be dyed in 7 colors. The number of ways to do this is
\[
\binom{7}{n-3} = \binom{n+3}{n-3} = \binom{n+3}{6}.
\]

(b) (5 points) ...the shirts are distinguishable?

**Solution:** The total number of ways to color the shirts is \( 7^n \). The number of ways for there to be no red shirt is \( 6^n \), and similarly for blue and green. The number of ways to avoid any two of the colors (which can be chosen in \( \binom{3}{2} = 3 \) ways) is \( 5^n \), and the number of ways to avoid all three colors is \( 4^n \). Hence by inclusion-exclusion, the answer is \( 7^n - 3 \cdot 6^n + 3 \cdot 5^n - 4^n \).
3. (10 points) In *The 85 Ways to Tie a Tie*, Thomas Fink and Yong Mao show that there are $a_n$ ways to tie a tie with $n$ or fewer turns, where the sequence

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 2, \quad a_3 = 5, \quad a_4 = 10, \quad a_5 = 21, \quad a_6 = 42, \quad a_7 = 85, \quad \ldots$$

is defined for $n \geq 1$ by

$$a_n = \begin{cases} 2a_{n-1} + 1, & \text{if } n \text{ is odd;} \\ 2a_{n-1}, & \text{if } n \text{ is even.} \end{cases}$$

Find a closed formula for $a_n$.

**Solution:** Let $A(x) = \sum_{n \geq 0} a_n x^n$. Then

$$(1 - 2x)A(x) = \sum_{n \geq 0} (a_n - 2a_{n-1})x^n = x + x^3 + x^5 + \cdots = \frac{x}{1 - x^2}.$$ 

Hence

$$A(x) = \frac{x}{(1 - 2x)(1 - x^2)} = \frac{c_1}{1 - 2x} + \frac{c_2}{1 - x} + \frac{c_3}{1 + x}.$$ 

To solve for the partial fractions, we clear denominators:

$$x = c_1(1 - x)(1 + x) + c_2(1 - 2x)(1 + x) + c_3(1 - 2x)(1 - x).$$

Now we solve:

- setting $x = \frac{1}{2}$ gives $\frac{1}{2} = c_1(\frac{1}{2})(\frac{3}{2})$, or $c_1 = \frac{2}{3}$;
- setting $x = 1$ gives $1 = c_2(-1)(2)$, or $c_2 = -\frac{1}{2}$;
- setting $x = -1$ gives $-1 = c_3(3)(2)$, or $c_3 = -\frac{1}{6}$.

Hence

$$A(x) = \frac{2/3}{1 - 2x} - \frac{1/2}{1 - x} - \frac{1/6}{1 + x} = \sum_{n \geq 0} \left(\frac{2}{3} \cdot 2^n - \frac{1}{2} \cdot 1^n - \frac{1}{6} \cdot (-1)^n\right).$$

It follows that $a_n = \frac{2}{3} \cdot 2^n - \frac{1}{2} - \frac{1}{6} \cdot (-1)^n$. 
4. Let \( n \geq 2 \) be a positive integer.

(a) (5 points) How many permutations of \([n]\) have exactly 2 inversions?

**Solution:** The number of permutations with 2 inversions is the coefficient of \( q^2 \) in \([n]_q!\). In other words, this is the number of tuples \((a_1, a_2, \cdots, a_n)\) such that \(0 \leq a_i < i\) and \(a_1 + a_2 + \cdots + a_n = 2\). The only way this can occur is if one of \(a_3, \ldots, a_n\) equals 2, or two of \(a_2, \ldots, a_n\) equal 1. Hence the answer is \((n-2) + \binom{n-1}{2} = \binom{n}{2} - 1\).

As a second solution, the number of permutations with 2 inversions equals the number of permutations with major index 2. This can only happen if there is exactly one descent at position 2. There are \(\binom{n}{2} - 1\) such permutations \(w\) (since there are \(\binom{n}{2}\) ways to choose \(w_1\) and \(w_2\), but we wish to exclude the identity).

As a third solution, any such permutation must be the product of two simple transpositions \(s_is_j\) with \(i \neq j\); note that \(s_i\) and \(s_j\) commute unless \(i = j \pm 1\). Hence there are \(\binom{n-1}{2}\) ways to choose \(i\) and \(j\), and \(n-2\) of those choices have \(i\) and \(j\) consecutive. This gives a total of \(\binom{n-1}{2} + (n-2) = \binom{n}{2} - 1\).

(b) (5 points) Show that the number of permutations of \([n]\) with exactly 2 cycles is

\[
(n-1)! \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1}\right).
\]

**Solution:** The answer is the Stirling number of the second kind \(c(n,2)\), which is the coefficient of \(t^2\) in \((t+1)(t+2)\cdots(t+n-1)\), which is the coefficient of \(t\) in \((t+1)(t+2)\cdots(t+n-1)\).

To find this coefficient, we must take the constant term from each factor save one. If we omit \(i\), then this contributes \((n-1)! \cdot \frac{1}{i^2}\). Summing over all \(i\) from 1 to \(n-1\) gives the result.

As a second solution, suppose the cycle that does not contain 1 has \(i\) elements. Then there are \(\binom{n-1}{i}\) ways to pick those elements, \((i-1)!\) ways to arrange them in a cycle, and \((n-i-1)!\) ways to arrange the remaining elements in a cycle (with 1). Multiplying gives a total of \((n-1)! \cdot \frac{1}{i^2}\) such permutations. Summing over all \(i\) from 1 to \(n-1\) gives the result.
5. (a) (5 points) For a fixed positive integer $k$, find the generating function $\sum_{n \geq 0} a_k(n)x^n$, where $a_k(n)$ is the number of partitions of $n$ with exactly $k$ parts, all of which are odd.

**Solution:** Let $\lambda$ be such a partition. Then the first column of $\lambda$ has length $k$, and if this column is removed, then all remaining column lengths are at most $k$ and must appear an even number of times. Hence for each $1 \leq i \leq k$, we must have a factor $1 + x^{2i} + x^{4i} + x^{6i} + \cdots = \frac{1}{1-x^{2i}}$. Thus the answer is

$$\sum_{n \geq 0} a_k(n)x^n = \frac{x^k}{(1-x^2)(1-x^4) \cdots (1-x^{2k})}.$$

(b) (5 points) A *bracelet* is an arrangement of beads in a cycle up to rotation and/or reflection. For instance, the following two bracelets are equivalent:

```
3  7
\circ \circ \circ
8  4
```

Let $a_n$ be the number of ways to form any number of bracelets, each with at least three beads, out of $n$ (distinguishable) beads total. (The order of the bracelets does not matter.) Find the exponential generating function $\sum_{n \geq 0} a_n \frac{x^n}{n!}$.

**Solution:** The number of ways to form a bracelet with $n \geq 3$ beads is $\frac{1}{2}(n-1)!$. This has exponential generating function

$$g(x) = \sum_{n \geq 3} \frac{1}{2}(n-1)! \cdot \frac{x^n}{n!} = \frac{1}{2} \sum_{n \geq 3} \frac{x^n}{n} = \frac{1}{2} \left( \log \left( \frac{1}{1-x} \right) - x - \frac{x^2}{2} \right).$$

Since we want to divide the beads into sets and form a bracelet from each set, the answer is

$$e^{g(x)} = \exp \left( \log \left( \frac{1}{\sqrt{1-x}} \right) - \frac{x}{2} - \frac{x^2}{4} \right) = \frac{e^{-\frac{x}{2} - \frac{x^2}{4}}}{\sqrt{1-x}}.$$