1. (a) Let \( a_n \) be the number of ordered tuples \((a, b, c, d)\) of integers satisfying
\[
0 \leq a < b \leq c < d \leq n.
\]
Find a closed formula for \( a_n \), as well as its ordinary generating function \( A(x) \).

**Solution 1:** Note that the given inequalities are equivalent to
\[
1 \leq a + 1 < b + 1 < c + 2 < d + 2 \leq n + 2.
\]
Hence these tuples are in bijection with 4-element subset of \([n + 2]\), so the answer is \( a_n = \binom{n+2}{4} \).

The corresponding generating function is
\[
A(x) = \sum_{n \geq 0} \binom{n+2}{4} x^n = \sum_{n \geq 0} \binom{n+4}{4} x^{n+2} = \frac{x^2}{(1-x)^5}.
\]

**Solution 2:** Note that \( a \geq 0, (b-a) \geq 1, (c-b) \geq 0, (d-c) \geq 1, \) and \( n-d \geq 0 \) are integers summing to \( n \). Hence the generating function for the number of ways to choose these elements is
\[
A(x) = \sum_{i \geq 0} x^i \cdot \sum_{i \geq 1} x^i \cdot \sum_{i \geq 0} x^i \cdot \sum_{i \geq 1} x^i \cdot \sum_{i \geq 0} x^i
\]
\[
= \frac{1}{1-x} \cdot \frac{x}{1-x} \cdot \frac{1}{1-x} \cdot \frac{x}{1-x} \cdot \frac{1}{1-x}
\]
\[
= \frac{x^2}{(1-x)^5}.
\]
Since
\[
\frac{x^2}{(1-x)^5} = \sum_{n \geq 0} \binom{n+4}{4} x^{n+2},
\]
it follows that \( a_n = \binom{n+2}{4} \).
(b) Let \( b_n \) be the number of ordered tuples \((A, B, C, D)\) of sets satisfying
\[
\emptyset \subseteq A \subsetneq B \subseteq C \subsetneq D \subseteq [n].
\]
Find a closed formula for \( b_n \), as well as its exponential generating function \( B(x) \).

**Solution 1:** We proceed by inclusion-exclusion. Note that the number of ways to choose \( A, B, C, \) and \( D \) without the restrictions that \( A \neq B \) and \( C \neq D \) is just \( 5^n \): each element of \([n]\) can lie in the largest 0, 1, 2, 3, or 4 of the sets. Similarly, if \( A = B \), then there are \( 4^n \) possibilities, and likewise if \( C = D \), while if \( A = B \) and \( C = D \), then there are \( 3^n \) possibilities. It follows that \( a_n = 5^n - 2 \cdot 4^n + 3^n \). The corresponding generating function is
\[
B(x) = \sum_{n \geq 0} (5^n - 2 \cdot 4^n + 3^n) \frac{x^n}{n!} = e^{5x} - 2e^{4x} + e^{3x}.
\]

**Solution 2:** The sets \( A, B \setminus A \neq \emptyset, C \setminus B, D \setminus C \neq \emptyset, \) and \([n] \setminus D\) partition \([n]\) into five disjoint subsets. Hence the exponential generating function for the number of ways to choose these sets is
\[
B(x) = \sum_{i \geq 0} \frac{x^i}{i!} \cdot \sum_{i \geq 1} \frac{x^i}{i!} \cdot \sum_{i \geq 0} \frac{x^i}{i!} \cdot \sum_{i \geq 1} \frac{x^i}{i!} \cdot \sum_{i \geq 0} \frac{x^i}{i!} = e^x \cdot (e^x - 1) \cdot e^x \cdot (e^x - 1) \cdot e^x = e^{5x} - 2e^{4x} + e^{3x}.
\]
Taking the coefficient of \( \frac{x^n}{n!} \) then gives \( b_n = 5^n - 2 \cdot 4^n + 3^n \).
2. How many partitions $\lambda$ (of any size $|\lambda| \geq 0$) satisfy $\lambda_i \leq n - i$ for all $i \leq n$?

**Solution:** This condition is equivalent to saying that the Young diagram fits inside a staircase shape of side length $n - 1$ (shown below for $n = 6$).

![Staircase Diagram]

But these are in bijection with Dyck paths of length $2n$: simply trace the boundary of the partition, starting one step below the southwest corner and ending one step to the right of the northeast corner. For instance, if $\lambda = 441$, we get the following path.

![Dyck Path Example]

It follows that the answer is the $n$th Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$.
3. Two standard decks of 52 playing cards are independently shuffled and placed side by side. Which is more likely: that no card is in the same position in both decks, or that there is exactly one such card?

Solution: Let $\sigma \in S_{52}$ be the permutation such that the $i$th card in the first deck is in position $\sigma(i)$ in the second deck. Then the number of cards in the same position in both decks is the number of fixed points of $\sigma$.

The number of permutations with no fixed points is the derangement number

$$D_{52} = 52! \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots - \frac{1}{51!} + \frac{1}{52!} \right).$$

The number of permutations with exactly one fixed point is $52 \cdot D_{51}$ since there are 52 ways to choose which card is fixed, and $D_{51}$ ways to permute the other cards without introducing any more fixed points. But

$$52 \cdot D_{51} = 52 \cdot 51! \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots - \frac{1}{51!} \right) = D_{52} - 1.$$

Hence it is (slightly, by $\frac{1}{52!}$) more likely for no card to be in the same position.
4. Give a combinatorial proof that
\[
c(n + 1, k + 1) = \sum_{i=k}^{n} \frac{n!}{i!} \cdot c(i, k),
\]
where \(c(n, k)\) denotes the Stirling number of the first kind.

**Solution:** The left side counts permutations of \([n + 1]\) with exactly \(k + 1\) cycles. We can alternatively count these as follows. Suppose 1 lies in a cycle with \(n - i\) other elements, say \((1 \ x_1 \ x_2 \ldots \ x_{n-i})\). Then there are
\[
n(n - 1)(n - 2) \cdots (i + 1) = \frac{n!}{i!}
\]
ways to choose this cycle (there are \(n\) possibilities for \(x_1\), \(n - 1\) possibilities for \(x_2\), and so on). The \(i\) remaining elements can then be divided into the remaining \(k\) cycles in \(c(i, k)\) ways. Since \(i\) can be any value from \(k\) to \(n\) (if \(i < k\), then there are not enough elements to split into \(k\) cycles), the result follows.
5. Let $a_n$ be the number of compositions of $n$ that do not contain a part of size exactly 2. For example, $a_5 = 7$:

$$5, \quad 41, \quad 14, \quad 311, \quad 131, \quad 113, \quad 11111.$$ 

Find a homogeneous linear recurrence with constant coefficients satisfied by $a_n$. (You do not need to supply the initial conditions.)

Solution 1: The generating function for $a_n$ is a composition $f(g(x))$, where $f(x) = \frac{1}{1-x}$ counts trivial “sequence” structures, and

$$g(x) = x + x^3 + x^4 + \cdots = \frac{x}{1-x} - x^2 = \frac{x - x^2 + x^3}{1-x}$$

counts “parts of size not equal to 2” structures. It follows that

$$\sum_{n \geq 0} a_n x^n = \frac{1}{1 - \frac{x - x^2 + x^3}{1-x}} = \frac{1-x}{1 - 2x + x^2 - x^3}.$$ 

Clearing the denominator gives

$$1 - x = (1 - 2x + x^2 - x^3) \cdot \sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} \left[(a_n - 2a_{n-1} + a_{n-2} - a_{n-3})x^n\right].$$

For $n \geq 2$, the coefficient of $x^n$ vanishes, so

$$a_n = 2a_{n-1} - a_{n-2} + a_{n-3}.$$

Solution 2: We will show that $a_n = 2a_{n-1} - a_{n-2} + a_{n-3}$, or equivalently,

$$b_n = b_{n-1} + a_{n-3},$$

where $b_n = a_n - a_{n-1}$.

To see this, note that we can attach a 1 onto the start of any composition of $n - 1$ to get a composition of $n$. Hence $b_n$ counts our desired compositions of $n$ that do not start with 1. In other words, they must start with a part of size at least 3. We can obtain such a composition in two ways:

- If the first part has size greater than 3, then we can subtract one from it, which yields a composition of size $n - 1$ whose first part has size at least 3. There are $b_{n-1}$ of these.
- If the first part has size exactly 3, then removing it yields a composition of size $n - 3$. There are $a_{n-3}$ of these.

Adding these together gives the desired recurrence.