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1. Let $L$ be a finite lattice.

(a) (5 points) Show that $p \land (q \lor (p \land r)) \geq (p \land q) \lor (p \land r)$ for all $p, q, r \in L$.

**Solution:**

- $p \land q \leq p$ and $p \land q \leq q \lor (p \land r)$ imply $p \land q \leq p \land (q \lor (p \land r))$.
- $p \land r \leq p$ and $p \land r \leq q \lor (p \land r)$ imply $p \land r \leq p \land (q \lor (p \land r))$.

Hence $(p \land q) \lor (p \land r) \leq p \land (q \lor (p \land r))$.

(b) (5 points) Show that equality holds if $L$ is modular.

**Solution:** If $L$ is modular, then it is graded with rank function $\rho$. By part (a), it suffices to show that the rank of the two sides is the same. Then

$$\rho(p \land (q \lor (p \land r))) = \rho(p) + \rho(q \lor (p \land r)) - \rho(p \lor q \lor (p \land r))$$

$$= \rho(p) + [\rho(q) + \rho(p \land r) - \rho(q \land p \land r)] - \rho(p \lor q)$$

$$= [\rho(p) + \rho(q) - \rho(p \lor q)] + \rho(p \land r) - \rho(q \land p \land r)$$

$$= \rho(p \land q) + \rho(p \land r) - \rho((p \land q) \land (p \land r))$$

$$= \rho((p \land q) \lor (p \land r)).$$

Here we used that

$$p \lor q \lor (p \land r) = (p \lor (p \land r)) \lor q = p \lor q$$

and

$$q \land p \land r = q \land p \land p \land r = (p \land q) \land (p \land r).$$
2. Let $P_n$ be the poset with elements $x_1, x_2, \ldots, x_n$ satisfying $x_1 > x_2 < x_3 > x_4 < x_5 > \cdots$ (with all other pairs of elements incomparable).

(a) (3 points) Draw the Hasse diagram of $J(P_5)$.

Solution:

(b) (3 points) How many linear extensions does $P_5$ have?

Solution: We wish to count permutations $w_1w_2w_3w_4w_5 \in S_5$ with $w_1 > w_2 < w_3 > w_4 < w_5$. Clearly either $w_2 = 1$ or $w_4 = 1$. If $w_2 = 1$, then there are 4 choices for $w_1$; then $w_4$ is the minimum remaining number and there are 2 choices for $w_3$ and $w_5$ for a total of 8 possibilities. Similarly, there are 8 possibilities if $w_4 = 1$ for a total of 16 linear extensions.

Alternatively, linear extensions of $P_5$ are in bijection with maximal chains in $J(P_5)$. We can find this by labeling the minimum element with 1 and then labeling each element with the sum of the labels of the elements it covers, as shown below.
(c) (4 points) Show that $|J(P_n)| = f_{n+2}$, where $f_n$ is the $n$th Fibonacci number ($f_1 = f_2 = 1$, $f_{n+2} = f_n + f_{n+1}$ for $n \geq 1$).

**Solution:** Since order ideals are in bijection with antichains, we will show that $P_n$ has $f_{n+2}$ antichains. Clearly $P_0$ has $f_2 = 1$ antichain ($\emptyset$) while $P_1$ has $f_3 = 2$ antichains ($\emptyset$ and $\{x_1\}$). For $n \geq 2$, any antichain $A$ contains either $x_n$ or does not. If it does not, then $A$ is one of the $f_{n+1}$ antichains in $P_{n-1}$. If it does, then it cannot contain $x_{n-1}$ (which is comparable to $x_n$), so the $A \setminus \{x_n\}$ is one of the $f_n$ antichains in $P_{n-2}$. Since $f_{n+2} = f_n + f_{n+1}$, the result follows.

3. (5 points) Suppose $\{a_n\}_{n \geq 1}$ satisfies, for all $n$,

$$\sum_{d|n} \frac{a_d}{d} = \frac{1}{n}.$$  

Find $a_{9000}$.

**Solution:** By Möbius inversion,

$$\frac{a_n}{n} = \sum_{d|n} \mu(d) \frac{1}{n/d} = \sum_{d|n} \frac{d}{n} \mu(d).$$

Hence $a_n = \sum_{d|n} d \mu(d)$. If $n$ has prime factorization $p_1^{q_1} \cdots p_r^{q_r}$, then $\mu(d)$ is only nonzero when $d$ is a product of $t$ distinct $p_i$, in which case $\mu(d) = (-1)^t$. Hence

$$a_n = 1 - \sum_i p_i + \sum_{i<j} p_ip_j - \sum_{i<j<k} p_ip_jp_k + \cdots = \prod_i (1 - p_i).$$

In particular, since $9000 = 2^33^25^3$, $a_{9000} = (1 - 2)(1 - 3)(1 - 5) = -8.$
4. Let $A$ be the hyperplane arrangement in $\mathbb{R}^4$ consisting of the four hyperplanes $x_1 = x_2, x_2 = x_3, x_3 = x_4, \text{ and } x_4 = x_1$.

(a) (3 points) Draw the Hasse diagram of the intersection poset $L(A)$.

**Solution:** Any pair of hyperplanes intersects in a distinct linear space of dimension 2: for instance, the first two intersect in the set where $x_1 = x_2 = x_3$, while the first and third intersect in the set where $x_1 = x_2$ and $x_3 = x_4$. However, all four hyperplanes contain the 1-dimensional subspace $x_1 = x_2 = x_3 = x_4$. Hence $L(A)$ looks as follows:

![Hasse diagram](image)

(b) (4 points) Find the characteristic polynomial $\chi_A(x)$.

**Solution:** By symmetry, $\mu(\hat{0}, t)$ for $t \in L(A)$ only depends on $\dim t$. We find that

$$
\mu(\hat{0}, t) = \begin{cases} 
1 & \text{if } \dim t = 4, \\
-1 & \text{if } \dim t = 3, \\
1 & \text{if } \dim t = 2, \\
-3 & \text{if } \dim t = 1.
\end{cases}
$$

(Note $\sum_{t \in L(A)} \mu(\hat{0}, t) = 1 \cdot 1 + 4 \cdot (-1) + 6 \cdot 1 + 1 \cdot (-3) = 0$.) Hence

$$
\chi_A(x) = \sum_{t \in L(A)} \mu(\hat{0}, t)x^{\dim t} = x^4 - 4x^3 + 6x^2 - 3x.
$$

(c) (3 points) Into how many regions does $A$ divide $\mathbb{R}^4$?

**Solution:** The number of regions of $A$ is $(-1)^4\chi_A(-1) = 1 + 4 + 6 + 3 = 14$. 
5. (5 points) Let $P = 2 \times 3 \times 4$, and let $\zeta \in I(P)$ be its zeta function. What is $\zeta^5(\hat{0}, \hat{1})$?

**Solution:** Recall that $\zeta^5(\hat{0}, \hat{1})$ counts the number of multichains of length 5 from $\hat{0}$ to $\hat{1}$, that is, the number of ways to choose $x_1, x_2, x_3, x_4 \in P$ such that $x_1 \leq x_2 \leq x_3 \leq x_4$. Since $P$ is a Cartesian product, it suffices to count the number of ways to make these choices in each factor and multiply the answers together.

In the chain $n$, there are $\left(\binom{n}{4}\right) = \binom{n+3}{4}$ ways to choose such a multichain. Hence the answer is

$$
\binom{5}{4} \binom{6}{4} \binom{7}{4} = 5 \cdot 15 \cdot 35 = 2625.
$$

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6. (5 points) Ten students are paired up for an assignment. For the next assignment, they must again pair up, but each person must have a different partner than before. In how many ways can this new pairing be made?

**Solution:** Note that the number of ways for $2k$ students to pair up is

$$(2k - 1)!! = (2k - 1)(2k - 3)(2k - 5) \cdots 3 \cdot 1.$$

Indeed, if we order the students, there are $2k - 1$ ways for the first student to choose a partner, then $2k - 3$ ways for the next unpaired student to choose, and so forth.

The number of ways in which a specific $m$ pairs can remain in the new pairing is then $(9 - 2m)!!$. Since there are $\binom{5}{m}$ ways to choose these $m$ pairs, by inclusion-exclusion, the number of ways for no pairs to be fixed is

$$
9!! - \binom{5}{1}7!! + \binom{5}{2}5!! - \binom{5}{3}3!! + \binom{5}{4}1!! - 1 = 945 - 5 \cdot 105 + 10 \cdot 15 - 10 \cdot 3 + 5 \cdot 1 - 1 = 544.
$$
7. Let \( n \) be a positive integer.

(a) (5 points) How many ways are there to write \( n \) as a sum of nonnegative integers \( a, b, c, \) and \( d \) if \( a \) must be odd and \( d \) must be 2 or 3?

Solution: The corresponding ordinary generating function is
\[
(x + x^3 + x^5 + \cdots)(1 + x + x^2 + x^3 + \cdots)^2(x^2 + x^3) = \frac{x}{(1 - x^2)^2} \frac{(1 + x^2)(1 - x)^2}{(1 - x)^3}
\]
\[
= \frac{x^3(1 + x)}{(1 + x)(1 - x)^3}
\]
\[
= \frac{x^3}{(1 - x)^3}
\]
\[
= \sum_{n \geq 3} \binom{n - 1}{2} x^n.
\]
Hence the answer is \( \binom{n - 1}{2} \).

(b) (5 points) How many ways are there to write \([n]\) as a disjoint union of sets \( A, B, C, \) and \( D \) if \(|A| \) must be odd and \(|D| \) must be 2 or 3?

Solution: The corresponding exponential generating function is
\[
\left( x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \right) \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right)^2 \left( \frac{x^2}{2!} + \frac{x^3}{3!} \right)
\]
\[
= \frac{1}{2}(e^x - e^{-x}) \cdot e^{2x} \cdot \left( \frac{x^2}{2} + \frac{x^3}{6} \right)
\]
\[
= \frac{1}{4}x^2(e^{3x} - e^x) + \frac{1}{12}x^3(e^{3x} - e^x)
\]
\[
= \frac{1}{4} \sum_{n \geq 2} (3^{n-2} - 1) \frac{x^n}{(n-2)!} + \frac{1}{12} \sum_{n \geq 3} (3^{n-3} - 1) \frac{x^n}{(n-3)!}.
\]
Hence the coefficient of \( \frac{x^n}{n!} \) is
\[
\frac{1}{4} (3^{n-2} - 1) \frac{n!}{(n-2)!} + \frac{1}{12} (3^{n-3} - 1) \frac{n!}{(n-3)!} = \binom{n}{2} \frac{3^{n-2} - 1}{2} + \binom{n}{3} \frac{3^{n-3} - 1}{2}.
\]
8. (5 points) Fix positive integers $k < n$. Let $G(k, n)$ be the set of Grassmannian permutations $w \in S_n$ satisfying $w_1 < w_2 < \cdots < w_k$ and $w_{k+1} < w_{k+2} < \cdots < w_n$.

Show that

$$
\sum_{w \in G(k, n)} q^{\text{inv}(w)} = \frac{n!}{k!}.
$$

**Solution:** For any $w \in G(k, n)$, note that

$$0 \leq w_1 - 1 < w_2 - 2 < \cdots < w_{k-1} - (k-1) < w_k - k \leq n - k.
$$

Hence we may associate to $w$ the partition

$$\lambda(w) = (w_k - k, \ w_{k-1} - (k-1), \ \cdots, \ w_2 - 2, \ w_1 - 1) \subseteq k \times (n-k).
$$

This map is a bijection: for any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \subseteq k \times (n-k)$, we define $w$ by $w_i = \lambda_{k+1-i} + i$ for $i \leq k$, and place all other $w_j$ (for $j > k$) in increasing order.

Moreover, the number of inversions of $w$ is $|\lambda(w)|$: the only possible inversions occur between $w_i$ for $i \leq k$ and $w_j$, for $j > k$, and any $w_i$ for $i \leq k$ is contained in exactly $w_i - i$ inversions (it is inverted with each element of $[w_i] \setminus \{w_{i'} \mid i' \leq i\}$). It follows that

$$
\sum_{w \in G(k, n)} q^{\text{inv}(w)} = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|} = \frac{n!}{k!}.
$$

**Solution:** Note that there is a bijection between $S_k \times S_{n-k} \times G(k, n)$ and $S_n$: given $u \in S_k$, $v \in S_{n-k}$ and $w \in G(k, n)$, we can let $u$ act on the first $k$ letters of $w$ and $v$ act on the last $n-k$ letters of $w$ to get a new permutation $z \in S_n$. To recover $(u, v, w)$ from $z$, simply sort the first $k$ letters of $z$ (which determines $u$) and the last $n-k$ letters of $z$ (which determines $v$) to get back $w$.

Note furthermore that $\text{inv}(z) = \text{inv}(u) + \text{inv}(v) + \text{inv}(w)$, since any inversion of $z$ either involves two letters among the first $k$, two letters among the last $n-k$, or one from each group, which correspond to inversions in $u$, $v$, or $w$, respectively. Hence,

$$
[n]_q! = \sum_{z \in S_n} q^{\text{inv}(z)}
= \sum_{u \in S_k, \ v \in S_{n-k}, \ w \in G(k, n)} q^{\text{inv}(u) + \text{inv}(v) + \text{inv}(w)}
= \sum_{u \in S_k} q^{\text{inv}(u)} \cdot \sum_{v \in S_{n-k}} q^{\text{inv}(v)} \cdot \sum_{w \in G(k, n)} q^{\text{inv}(w)}
= [k]_q ![n-k]_q! \cdot \sum_{w \in G(k, n)} q^{\text{inv}(w)}.
$$

Dividing both sides by $[k]_q ![n-k]_q!$ gives the result.
**Solution:** Any permutation $w \in G(k, n)$ has either $w_k = n$ or $w_n = n$.

- If $w_k = n$, then $w$ is obtained from a permutation $u \in G(k - 1, n - 1)$ by inserting $n$ into the $k$th position. Then $\text{inv}(w) = \text{inv}(u) + (n - k)$.

- If $w_n = n$, then $w$ is obtained from a permutation $v \in G(k, n - 1)$ by adding $n$ at the end. Then $\text{inv}(w) = \text{inv}(v)$.

Hence

$$
\sum_{w \in G(k, n)} q^{\text{inv}(w)} = q^{n-k} \sum_{u \in G(k-1, n-1)} q^{\text{inv}(u)} + \sum_{v \in G(k, n-1)} q^{\text{inv}(v)}.
$$

The result now follows easily by induction using the identity

$$
\binom{n}{k}_q = q^{n-k} \binom{n-1}{k-1}_q + \binom{n-1}{k}_q.
$$