Remember to justify your answers. The maximum possible score is 50 points.

1. Let $\mathcal{A}$ be the real hyperplane arrangement in $\mathbb{R}^2$ shown below.

(a) (5 pts) Draw the Hasse diagram of $L(\mathcal{A})$ and compute $\mu(\hat{0}, t)$ for each $t \in L(\mathcal{A})$.

**Solution:** The intersection poset together with the Möbius functions are shown below.

(b) (2 pts) What is the characteristic polynomial of $\mathcal{A}$?

**Solution:** The characteristic polynomial is

$$\chi_{\mathcal{A}}(x) = \sum_{t \in \mathcal{A}} \mu(\hat{0}, t)x^{\dim t} = x^2 - 5x + 6.$$  

As a check, note that $(-1)^2\chi_{\mathcal{A}}(-1) = 12$, the number of regions of $\mathcal{A}$, while $(-1)^2\chi_{\mathcal{A}}(1) = 2$, the number of bounded regions of $\mathcal{A}$. 
2. (2 pts each) For each of the following, draw the Hasse diagram of a finite poset satisfying the first condition but not the second. Briefly explain why your answer does not satisfy the second condition. (You do not need to explain why it satisfies the first condition.)

(a) A poset with a minimum and a maximum but not a lattice

**Solution:** This is not a lattice since \(x\) and \(y\) have no least upper bound.

(b) A lattice but not a semimodular lattice

**Solution:** This is not semimodular since (for instance) it is not graded. (Alternatively, \(x\) and \(y\) cover \(\hat{0}\) but they are not covered by any element.)

(c) A semimodular lattice but not a modular lattice

**Solution:** This is not modular since \(x\) and \(y\) are covered by \(\hat{1}\) but do not cover any element.

(d) A modular lattice but not a distributive lattice

**Solution:** This is not distributive since \(x \lor (y \land z) = x \neq \hat{1} = (x \lor y) \land (x \lor z)\). (Alternatively, its join-irreducibles are \(x\), \(y\), and \(z\), which form an antichain of size 3, but the poset is not the Boolean lattice \(B_3\).)
3. (5 pts) Let $L$ be a lattice. Show that for all $a, b, c \in L$,

$$(a \land b) \lor (b \land c) \lor (c \land a) \leq (a \lor b) \land (b \lor c) \land (c \lor a).$$

**Solution:** Note that

- $a \land b \leq a$ and $a \land b \leq b$ by definition of meet.
- $a \leq a \lor b$, $b \leq b \lor c$, and $a \leq c \lor a$ by definition of join.

Hence by transitivity, $a \land b$ is less than or equal to $a \lor b$, $b \lor c$, and $c \lor a$, so it is less than or equal to their meet, which is the righthand side. A similar argument shows that $b \land c$ and $c \land a$ are also less than or equal to the righthand side, so their join $(a \land b) \lor (b \land c) \lor (c \land a)$ is also.
4. (5 pts) Let $n$ be a positive integer. Define the poset $\Phi = \{\alpha_{ij} \mid 1 \leq i \leq j \leq n\}$ with partial order $\alpha_{ij} \preceq \alpha_{kl}$ if $k \leq i \leq j \leq l$. How many antichains does $\Phi$ have?

**Solution:** The poset $\Phi$ is shaped like a triangle, shown below for $n = 6$.

Antichains in any poset are in bijection with its order ideals, so we need only count the order ideals of $\Phi$.

We claim that the answer is the Catalan number $C_{n+1} = \frac{1}{n+2} \binom{2n+2}{n+1}$. Indeed, we can embed $\Phi$ in the plane by placing $\alpha_{ij}$ at the point $(i + j, j - i + 1)$.

The points of $\Phi$ that lie below any Dyck path from $(0, 0)$ to $(2n + 2, 0)$ will then be an order ideal $I$ of $\Phi$. Conversely, the path that separates $I$ from $\Phi \setminus I$ forms a Dyck path. (Equivalently, an antichain of $\Phi$ marks the “peaks” above height 1 of some Dyck path.)
5. (5 pts) Consider the set $S$ of all lattice paths from $(0, 0)$ to $(8, 8)$ consisting of unit steps north and east that stay within the region $|x - y| \leq 5$. Describe an explicit bijection between $S$ and the set of linear extensions of some poset $P$. (Do not compute the size of $S$.)

**Solution:** Rotated 45° degrees, the elements of $S$ correspond to maximal chains in the following distributive lattice $L$ (where every intersection point is an element of $L$). Since $L$ is distributive, we can write $L \cong J(P)$, where $P$ is the poset of join-irreducibles (marked with circles).

![Distributive Lattice](image)

The maximal chains of a distributive lattice are in bijection with the linear extensions of its poset of join-irreducibles, so it follows that $S$ is in bijection with linear extensions of $P$ (reproduced below).

![Linear Extensions](image)

Explicitly, for any linear extension $\sigma: P \to [16]$, let the $i$th step of the walk be to the north if $\sigma^{-1}(i)$ lies in the left chain and to the east if $\sigma^{-1}(i)$ lies in the right chain. (The “crossing” edges prevent the walk from stepping outside the desired region.)
6. (5 pts) How many coefficients in the expansion of \((x + y + z + w)^{11}\) are divisible by 11?

**Solution:** Since each monomial in the expansion has the form \(x^a y^b z^c w^d\) for some \(a, b, c, d \geq 0\) with \(a + b + c + d = 11\), the total number of terms is just the number of such tuples \((a, b, c, d)\), which is (using a stars-and-bars argument)

\[
\binom{11 + 4 - 1}{3} = \binom{14}{3} = \frac{14 \cdot 13 \cdot 12}{3!} = 364.
\]

The coefficient of this monomial, by the multinomial theorem, is

\[
\binom{11}{a, b, c, d} = \frac{11!}{a!b!c!d!},
\]

which will be divisible by 11 whenever none of \(a, b, c, d\) equal 11. (If any of them equal 11, then this coefficient will be 1.) Hence all but 4, or \(364 - 4 = 360\) coefficients are divisible by 11.
7. Suppose bananas cost $1 each, while apples and pears cost $2 each. (Assume two pieces of fruit of the same type are indistinguishable.)

(a) (5 pts) Let $a_n$ be the number of ways to pick out exactly $n$ dollars worth of fruit if the order in which the fruit is chosen does not matter. Find the generating function $\sum_{n\geq 0} a_n x^n$, and calculate $a_{100}$.

Solution: The generating function for the number of ways to choose $n$ dollars worth of bananas is $\frac{1}{1-x} = \sum_{n\geq 0} x^n$, while that for either apples or pears is $\frac{1}{1-x} = \sum_{n\geq 0} x^{2n}$. The generating function for all fruit is then the product

$$\frac{1}{1-x} \cdot \frac{1}{1-x^2} = \sum_{n\geq 0} x^{2n} = \frac{1}{(1-x)^3(1+x)^2} = \frac{1+x}{(1-x^2)^3}.$$ 

Since

$$\frac{1+x}{(1-x^2)^3} = (1+x) \sum_{n\geq 0} \binom{n+2}{2} x^{2n} = \sum_{n\geq 0} \binom{n+2}{2} (x^{2n} + x^{2n+1}),$$

the coefficient of $x^{100}$ is $\binom{52}{2} = 1326$.

(b) (5 pts) Let $b_n$ be the number of ways to pick out exactly $n$ dollars worth of fruit if the order in which the fruit is chosen does matter. Find the generating function $\sum_{n\geq 0} b_n x^n$, and calculate $b_{100}$.

Solution: Note that for $n \geq 2$, $b_n = b_{n-1} + 2b_{n-2}$ since we can either choose a banana first and then $n-1$ dollars worth of fruit, or we can choose an apple or pear first and then $n-2$ dollars worth of fruit. Together with $b_0 = b_1 = 1$, it follows that

$$(1 - x - 2x^2) \sum_{n\geq 0} b_n x^n = \sum_{n\geq 0} (b_n - b_{n-1} - b_{n-2}) x^n = 1,$$

so the generating function is

$$\frac{1}{1-x-2x^2} = \frac{1}{(1-2x)(1+x)} = \frac{2/3}{1-2x} + \frac{1/3}{1+x}.$$ 

It follows that $b_n = \frac{1}{3} (2^{n+1} + (-1)^n)$, so $b_{100} = \frac{1}{3}(2^{101} + 1)$.
8. (5 pts) Let $d(n,k)$ be the number of derangements of $[n]$ with exactly $k$ cycles. Express $d(n,k)$ as a sum involving Stirling numbers of the first kind $c(n,k)$.

**Solution:** There are $c(n,k)$ total permutations with $k$ cycles. Of these, $c(n-1, k-1)$ of them have 1 as a fixed point, and similarly for the other $n-1$ possible fixed points. Likewise, $c(n-2, k-2)$ of them have 1 and 2 as a fixed point, and similarly for the other ${n-1 \choose 2} - 1$ ways of choosing two fixed points. Similarly, for any of the ${n \choose i}$ subsets $S$ of size $i$, $c(n-i, k-i)$ of them have all numbers in $S$ as fixed points. Thus, by inclusion-exclusion, the number with no fixed points is

$$d(n,k) = \sum_{i=0}^{n} (-1)^i {n \choose i} c(n-i, k-i).$$