MA 524 Homework 1 – Solutions

1. Here are three algebraic proofs:

- By the binomial theorem,
  \[ \sum_{k=0}^{n} \binom{n}{k} x^k = (1 + x)^n. \]

  Taking the derivative with respect to \( x \) gives
  \[ \sum_{k=1}^{n} k \binom{n}{k} x^{k-1} = n(1 + x)^{n-1}. \]

  Plugging in \( x = 1 \) then gives the result.

- We can calculate
  \[ k \binom{n}{k} = \frac{k \cdot n!}{k!(n-k)!} = \frac{n \cdot (n-1)!}{(n-k)!(k-1)!} = n \binom{n-1}{k-1}. \]

  Hence
  \[ \sum_{k=0}^{n} k \binom{n}{k} = n \sum_{k=1}^{n} \binom{n-1}{k-1} = n 2^{n-1}. \]

- Since \( \binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k} \), reversing the order of summation gives
  \[ \sum_{k=0}^{n} k \binom{n}{k} = \sum_{k=0}^{n} (n-k) \binom{n}{n-k} = \sum_{k=0}^{n} (n-k) \binom{n}{k}. \]

  Hence
  \[ \sum_{k=0}^{n} k \binom{n}{k} = \frac{1}{2} \left( \sum_{k=0}^{n} k \binom{n}{k} + \sum_{k=0}^{n} (n-k) \binom{n}{k} \right) \]
  \[ = \frac{1}{2} \cdot n \cdot \sum_{k=0}^{n} \binom{n}{k} \]
  \[ = \frac{1}{2} n \cdot 2^n \]
  \[ = n 2^{n-1}. \]

For a combinatorial proof, the left side counts the number of ways to choose any subset \( S \subseteq [n] \) (of size \( k \) for \( 0 \leq k \leq n \)) and then choose a special element \( s \in S \) (in \( k \) ways).

We could alternatively first choose which \( s \in [n] \) is special in \( n \) ways and then choose the rest of \( S \) as a subset of \([n] \setminus \{s\} \) in \( 2^{n-1} \) ways.
2. The simplest proof is the following algebraic proof:

- Suppose we wish to compute the coefficient of \(x^n\) in \((1 - x)^n(1 + x)^n\). We must multiply the coefficient of \(x^k\) from \((1 - x)^n\) with the coefficient of \(x^{n-k}\) from \((1 + x)^n\), and then sum over all values of \(k = 0, 1, \ldots, n\). Hence, by the binomial theorem, this coefficient is

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \cdot \binom{n}{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2,
\]

which is the sum we wish to find. But

\[
(1 - x)^n(1 + x)^n = (1 - x^2)^n = \sum_{k=0}^{n} \binom{n}{k} (-x^2)^k = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{2k}.
\]

Thus the coefficient of \(x^n\) is \(\binom{n}{n/2}(-1)^{n/2}\) if \(n\) is even (taking \(k = \frac{n}{2}\)) and 0 if \(n\) is odd (since there are no terms with odd exponent in the sum).

A combinatorial proof is tricky but also possible:

- Let \(X\) be the set of all pairs of subsets \(A, B \subseteq [n]\) such that \(|A| = |B|\), and associate to each such pair a sign \(w(A, B) = (-1)^{|A|}\). The sum of the signs of all elements in \(X\) is then the sum we wish to find. We construct an involution \(\tau: X \to X\) as follows.

Let \(i\) be the smallest element that lies in either neither or both of \(A\) and \(B\) (if such an element exists). Then let \(\tau(A, B) = (A', B')\), where either \(A' = A \cup \{i\}\) and \(B' = B \cup \{i\}\) if \(i \notin A, B\), or else \(A' = A \setminus \{i\}\) and \(B' = B \setminus \{i\}\) if \(i \in A, B\). (If no such \(i\) exists, then set \(\tau(A, B) = (A, B)\).) Clearly \(\tau^{-1} = \tau\) (since it just adds or removes \(i\) again as appropriate).

Since \(A'\) and \(A\) differ in size by 1, \((A, B)\) and \((A', B')\) have opposite sign, so \(\tau\) pairs up elements of \(X\) of opposite sign. Therefore if we want the sum of the weights of elements in \(X\), elements paired up by \(\tau\) cancel out, so all that remains are the elements fixed by \(\tau\). Thus we just need to find the total weight of these fixed points, which only occur when \(A\) and \(B\) have size \(\frac{n}{2}\) and are complementary.

There are \(\binom{n}{n/2}\) such pairs with sign \((-1)^{n/2}\) if \(n\) is even, and no such pairs if \(n\) is odd.

3. For each \(i \in [n]\), there are 7 possibilities for which of \(A, B, C,\) and \(D\) it can lie in: it can either lie in none of the four sets, in exactly one of the four sets, in \(A\) and \(C\) but not \(B\) and \(D\), or in \(B\) and \(D\) but not \(A\) and \(C\). Deciding which sets each \(i \in [n]\) lie in determines the four sets exactly, and these choices are independent. It follows that the answer is \(7^n\).

It is also possible to give an algebraic proof, for instance, by using the fact that the condition is equivalent to \((A \cup C) \cap (B \cup D) = \emptyset\) to write down a complicated summation formula and then using the binomial theorem repeatedly.
4. Let $S$ be the set of $i$ for which the $i$th step is either up or right, and let $T$ be the set of $i$ for which the $i$th step is either up or left. Note that since there must be the same number of steps up as down and right as left, both $S$ and $T$ have size $n$. We claim that this gives a bijection between valid walks and pairs $(S, T)$ of $n$-element subsets of $[2n]$, of which there are $\binom{2n}{n}^2$.

Indeed, given $(S, T)$, consider the walk whose $i$th step is up if $i \in S$ and $i \in T$, right if $i \in S$ but $i \not\in T$, left if $i \in T$ but $i \not\in S$, and down if $i \not\in S$ and $i \not\in T$. This walk will always end at the origin because the numbers of east steps and west steps are both $n - |S \cap T|$, while the numbers of north steps and south steps are both $|S \cap T|$. It is now easy to see that this is the inverse of the map above, so we have the desired bijection.

For an algebraic proof, let $i$ be the number of left steps. Then the total number of walks is the number of ways to order $i$ left steps, $i$ right steps, $n - i$ up steps, and $n - i$ down steps, which is

$$\sum_{i=0}^{n} \frac{(2n)!}{i!(n-i)!} = \sum_{i=0}^{n} \frac{(2n)!}{n!i!(n-i)!} \frac{n!}{n!} = \left(\frac{2n}{n}\right) \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array}\right)^{2} = \left(\frac{2n}{n}\right)^{2}$$

by Vandermonde’s identity.

5. Let $\omega = e^{2\pi i/3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ so that $\omega^3 = 1$. Consider for any integer $k$ the quantity $1 + \omega^k + \omega^{2k}$. If $k$ is a multiple of 3, then each term equals 1, so this equals 3. If $k$ is not a multiple of 3, then it equals $(1 - \omega^{3k})/(1 - \omega^k) = 0$. Therefore, for any polynomial $f(x) = \sum_{k} a_k x^k$,

$$f(1) + f(\omega) + f(\omega^2) = \sum_{k} a_k (1 + \omega^k + \omega^{2k}) = 3(a_0 + a_3 + a_6 + \cdots).$$

Now let $f(x) = (1 + x)^n$. Then $a_k = \left(\begin{array}{c} n \\ k \end{array}\right)$ by the binomial theorem, so by the above,

$$\left(\begin{array}{c} n \\ 0 \end{array}\right) + \left(\begin{array}{c} n \\ 3 \end{array}\right) + \left(\begin{array}{c} n \\ 6 \end{array}\right) + \cdots = \frac{1}{3} \left((1 + 1)^n + (1 + \omega)^n + (1 + \omega^2)^n\right)$$

$$= \frac{1}{3} \left(2^n + \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^n + \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^n\right)$$

$$= \frac{1}{3} \left(2^n + e^{n\pi i/3} + e^{-n\pi i/3}\right)$$

$$= \frac{1}{3} \left(2^n + 2 \cos \frac{n\pi}{3}\right).$$

Alternatively, one can also give an inductive proof, provided one tries to prove the stronger statement that the number of subsets $S$ such that $|S| \equiv m \pmod{3}$ is

$$f_m(n) = \frac{1}{3} \left(2^n + 2 \cos \frac{(n - 2m)\pi}{3}\right).$$
The base case $n = 0$ is easy to verify directly. By Pascal’s recurrence,

$$f_m(n) = \sum_k \binom{n}{3k + m} = \sum_k \left( \binom{n-1}{3k + m} + \binom{n-1}{3k + m - 1} \right) = f_m(n-1) + f_{m-1}(n-1).$$

But by the inductive hypothesis, this sum is

$$\frac{1}{3} \left( 2^{n-1} + 2 \cos \frac{(n-2m-1)\pi}{3} + 2^{n-1} + 2 \cos \frac{(n-2m+1)\pi}{3} \right).$$

It therefore suffices to check that

$$\cos \frac{(n-2m-1)\pi}{3} + \cos \frac{(n-2m+1)\pi}{3} = \cos \frac{(n-2m)\pi}{3},$$

which is a special case of the trigonometric identity

$$\cos(x - y) + \cos(x + y) = 2 \cos x \cos y$$

when $x = \frac{(n-2m)\pi}{3}$ and $y = \frac{\pi}{3}$. 

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