MA 524 Homework 3 – due 9/13

Homework:

1. Let \( n \geq 4 \) be an even integer.

   (a) How many compositions of \( n \) have four parts, all of which are odd?

   (b) How many ways are there to partition \([n]\) into four disjoint subsets \(A, B, C,\) and \(D\), all of which have odd size?

2. Show that

\[
\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} = 4^n.
\]

3. (a) Let \( M_n \) be the number of lattice paths from \((0,0)\) to \((n,0)\) using steps of the form \((1,1), (1,-1),\) or \((1,0)\) that do not pass below the \(x\)-axis. The sequence \(\{M_n\}_{n \geq 0}\) begins

\[
1, 1, 2, 4, 9, 21, 51, 127, \ldots
\]

Find the generating function for \(M_n\).

   (b) Let \( T_n \) be the coefficient of \(x^n\) in \((1 + x + x^2)^n\). The sequence \(\{T_n\}_{n \geq 0}\) begins

\[
1, 1, 3, 7, 19, 51, 141, 393, \ldots
\]

Find the generating function for \(T_n\).

4. (a) Show that the Catalan number \(C_n = \frac{1}{n+1} \binom{2n}{n}\) counts the number of ways to draw \(n\) nonintersecting chords in a circle given the \(2n\) endpoints by giving a bijection to Dyck paths of length \(2n\).

   (b) How many ways are there to draw any number of nonintersecting chords among \(n\) points on a circle?