MA 524 Homework 4 – due 9/25

Homework:

1. Let \( a_n \) be the number of orders in which \( n \) people can finish a race if ties are allowed. (The sequence \( \{a_n\}_{n \geq 0} \) begins 1, 1, 3, 13, 75, . . . .) Find the exponential generating function \( f(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!} \).

2. Consider all Dyck paths of length \( 2n \) (counted by the Catalan number \( C_n = \frac{1}{n+1}\binom{2n}{n} \)). How many total times do they touch the line \( y = x \)? (For example, when \( n = 2 \), there are two Dyck paths which touch two and three times, respectively, so the total number of touches is 5.)

3. Let \( n \) be a positive integer. How many permutations in \( S_n \) have only cycles of odd length?

4. Let \( n \geq 2 \) be a positive integer. Do more derangements in \( S_n \) have an even number of cycles or an odd number of cycles?

5. For positive integers \( k \leq n \), give both a combinatorial and an algebraic proof that

\[
\sum_{i=k}^{n} c(n, i) \binom{i}{k} = c(n + 1, k + 1).
\]