MA 524 Homework 4 – Solutions

Homework:

1. To form such an order, we first split the $n$ people into nonempty subsets, and then we arrange these subsets into a linear order. Hence, $f(x) = g(h(x))$, where $g(x)$ and $h(x)$ are the exponential generating functions for linear orders and nonempty set structures, respectively. Since $g(x) = \sum_{n \geq 0} \frac{n!}{n!} x^n = \frac{1}{1-x}$ and $h(x) = \sum_{n \geq 1} \frac{x^n}{n!} = e^x - 1$,

$$f(x) = g(h(x)) = \frac{1}{1 - (e^x - 1)} = \frac{1}{2 - e^x}.$$  

One can also find this by writing down a recurrence for $a_n$. Suppose there are $k$ people in the group that finishes first. Then there are $\binom{n}{k}$ ways to choose these people and $a_{n-k}$ ways in which the rest of the people can finish. Hence

$$f(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!} = 1 + \sum_{n \geq 1} \sum_{k=1}^{n} \binom{n}{k} a_{n-k} \frac{x^n}{k!} = 1 + \sum_{k \geq 1} \frac{x^k}{k!} \cdot f(x) = 1 + (e^x - 1) f(x),$$

which easily gives the result.

2. Algebraically, if a Dyck path consists of a sequence of $k$ nonreturning Dyck paths, then it touches the diagonal $k+1$ times. Hence the generating function for touches is $f(g(x))$, where $f(x) = \sum_{k \geq 0} (k+1)x^k = \frac{1}{(1-x)^2}$, and $g(x)$ is the generating function for nonreturning Dyck paths.

Recall that $g(x) = xC(x)$, where $C(x)$ is the generating function for Dyck paths, which satisfies $xC(x)^2 = C(x) - 1$, or $\frac{1}{1-xC(x)} = C(x)$. Hence

$$f(g(x)) = \frac{1}{(1-xC(x))^2} = C(x)^2 = \frac{C(x) - 1}{x} = \sum_{n \geq 0} C_{n+1} x^n,$$

so there are $C_{n+1} = \frac{1}{n+2} \binom{2n+2}{n+1}$ touches.

Alternatively, Dyck paths with one touch marked are in bijection with pairs of Dyck paths whose lengths sum to $2n$ (where the marked touch is the one between the two paths); hence the desired generating function is $C(x)^2$, as above.

Combinatorially, note that any Dyck path of length $2n+2$ consists of a nonreturning Dyck path followed by an ordinary Dyck path. If we remove the first and last step from the nonreturning Dyck path and mark its endpoint, the result is a Dyck path of length $2n$ with one of its touches marked. Conversely, given a Dyck path of length $2n$ with one of its touches marked, we can elevate the part of the Dyck path before that touch by adding an up step at its start and a down step at its end, and the result is a Dyck path of length $2n + 2$. It is clear that these bijections are inverses, and the result follows.
3. To find the generating function for the answer, we should take the generating function for the cycle index polynomial and set $t_n = 0$ if $n$ is even and 1 if $n$ is odd. This gives

$$
\exp \left( x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \right) = \exp \left[ \frac{1}{2} \left( \sum_{n \geq 0} \frac{x^n}{n} - \sum_{n \geq 0} \frac{(-x)^n}{n} \right) \right]
= \exp \left[ \frac{1}{2} \left( \log \frac{1}{1-x} - \log \frac{1}{1+x} \right) \right]
= \exp \left( \log \sqrt{\frac{1+x}{1-x}} \right)
= \sqrt{\frac{1+x}{1-x}}
= \frac{1+x}{\sqrt{1-x^2}}.
$$

Using the binomial theorem, we can calculate

$$
\frac{1}{\sqrt{1-x^2}} = \sum_{k \geq 0} \binom{-\frac{1}{2}}{k} (-x^2)^k
= \sum_{k \geq 0} \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdots \frac{2k-1}{2}}{k!} x^{2k}
= \sum_{k \geq 0} \frac{(2k-1)!!}{2^k k!} x^{2k}
= \sum_{k \geq 0} \frac{(2k-1)!!}{(2k)!} \cdot \frac{x^{2k}}{(2k)!}.
$$

(Here we use the fact that $(2k-1)!! = (2k-1)(2k-3) \cdots 3 \cdot 1 = \frac{(2k)!}{2^k k!}$. ) This generating function has only even exponents, so

$$
[x^{2k}] \frac{1+x}{\sqrt{1-x^2}} = [x^{2k+1}] \frac{1+x}{\sqrt{1-x^2}} = [x^{2k}] \frac{1}{\sqrt{1-x^2}} = \frac{(2k-1)!!}{(2k)!}.
$$

To find the answer, we then multiply this coefficient by either $(2k)!$ or $(2k+1)!$ depending on whether $n = 2k$ is even or $n = 2k + 1$ is odd. We then find for $n$ even, there are $(n-1)!!^2$ such permutations, while if $n$ is odd, then there are $(n-2)!!^2 \cdot n = n!! \cdot (n-2)!!$ such permutations.

Surprisingly, it turns out that for $n$ even, a similar calculation shows that the number of permutations with only cycles of even length is again $(n-1)!!^2$.

4. If we want the difference between the number of derangements with an even number of cycles and the number of derangements with an odd number of cycles, then we should
set \( t_1 = 0 \) and all other \( t_n = -1 \) in the cycle index polynomial. (Any permutation with a fixed point will not get counted, otherwise it will get counted by either 1 or \(-1\) depending on whether the number of cycles is even or odd.)

We therefore get the generating function

\[
\exp \left( \sum_{n \geq 1} -\frac{x^n}{n} \right) = \exp \left( -\log \frac{1}{1-x} + x \right) = \exp(\log(1-x) + x) = (1-x)e^x.
\]

Now

\[
(1-x)e^x = e^x - xe^x = \sum_{n \geq 0} \frac{x^n}{n!} - \sum_{n \geq 1} \frac{x^n}{(n-1)!} = \sum_{n \geq 0} (1-n)\frac{x^n}{n!}.
\]

Hence there are \( n-1 \) more derangements with an odd number of cycles than with an even number of cycles.

5. Algebraically, recall that \( \sum_{i=0}^{n} c(n,i)t^i = t(t+1)(t+2) \cdots (t+n-1) \). Then

\[
\sum_{i=0}^{n} c(n,i)(t+1)^i = (t+1)(t+2) \cdots (t+n-1)(t+n)
\]

\[
= t^{-1} \sum_{i=1}^{n+1} c(n+1,i)t^i
\]

\[
= \sum_{i=0}^{n} c(n+1,i+1)t^i
\]

Now extracting the coefficient of \( t^k \) on both sides gives

\[
\sum_{i=k}^{n} c(n,i)\left(\begin{array}{c}i \\ k\end{array}\right) = c(n+1,k+1).
\]

Combinatorially, the right side counts permutations \( w \) of \([n+1]\) with \( k+1 \) cycles.

Let \( A = \{a_1 < a_2 < \cdots < a_m\} \) be the set of elements in the same cycle as \( n+1 \), and suppose this cycle is \( (n+1 \ \alpha(a_1) \ \alpha(a_2) \ \cdots \ \alpha(a_m)) \) for some permutation \( \alpha \) of \( A \). Then we can construct a permutation \( v \) of \([n]\) by letting \( v(a_i) = \alpha(a_i) \) if \( a_i \in A \) and \( v(x) = w(x) \) otherwise. If we mark the \( k \) cycles containing elements not in \( A \), this yields a permutation of \([n]\) with \( k \) distinguished cycles, of which there are \( \sum_{i=k}^{n} c(n,i)\left(\begin{array}{c}i \\ k\end{array}\right) \).

Conversely, given a permutation \( v \) of \([n]\) with \( k \) distinguished cycles, we can construct \( w \) by letting \( A = \{a_1 < a_2 < \cdots < a_m\} \) be the set of elements not in any distinguished cycle and letting \( w \) contain all the distinguished cycles of \( v \) together with the cycle \( (n+1 \ v(a_1) \ v(a_2) \ \cdots \ v(a_m)) \). This is clearly the inverse to the map above, giving the desired bijection.