1. We prove the first claim; the second follows by duality. Note that \( a \wedge b \leq a \), and \( a \wedge b \leq b \leq b \vee c \). Hence \( a \wedge b \leq a \wedge (b \vee c) \). The same argument with \( b \) and \( c \) switched gives \( a \wedge c \leq a \wedge (c \vee b) = a \wedge (b \vee c) \). Hence \( a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c) \).

2. Consider the set \( S = \{ x \in L \mid x \leq f(x) \} \). Note that \( S \) is nonempty since \( \hat{0} \in S \). Let \( p = \bigvee_{x \in S} x \). We claim that \( p \) is the greatest fixed point of \( f \). Since all fixed points lie in \( S \), it suffices to show that \( p \) is itself a fixed point.

To see this, note that since \( f \) is order-preserving and \( p \geq x \) for all \( x \in S \), we have \( f(p) \geq f(x) \geq x \) for all \( x \in S \). Thus \( f(p) \geq \bigvee_{x \in S} x = p \). Now, since \( f \) is order-preserving and \( p \leq f(p) \), we have \( f(p) \leq f(f(p)) \). But this implies \( f(p) \in S \), so \( f(p) \leq \bigvee_{x \in S} x = p \). It follows that \( p = f(p) \), so \( p \) is a fixed point.

Here is an alternate proof when \( L \) is finite. We first show that any order-preserving map \( f \) from a finite lattice to itself has a fixed point. For any element \( x \in L \), we have that \( x \leq f(x) \leq f(f(x)) \leq f(f(f(x))) \leq \cdots \) since \( f \) is order-preserving. Since \( L \) is finite, this sequence must stabilize at some point, which implies that some \( y = f(f(\cdots f(x)\cdots)) \) is a fixed point of \( f \). Now let \( S \) be the set of fixed points of \( f \), which is nonempty, and let \( p \) be the join of all these fixed points. As in the previous argument, \( f(p) \geq f(x) = x \) for all \( x \in S \), so \( f(p) \geq \bigvee_{x \in S} x = p \). But now the interval \( I = [p, \hat{1}] \) is also a lattice, and the restriction \( f|_I \) sends any \( z \in I \) to an element \( f(z) \geq f(p) \geq p \), so \( f(z) \in I \). In other words, \( f|_I \) is an order-preserving map on the interval \( I \), so it must have a fixed point. But any fixed point of \( f|_I \) is also a fixed point of \( f \), all of which are at most \( p \). It follows that \( p \) must be a fixed point, as desired.

3. For the reverse direction, the condition implies that \( u \) covers \( u \wedge v \) if and only if \( u \vee v \) covers \( v \), and similarly if we switch \( u \) and \( v \). This immediately implies that \( u \) and \( v \) cover \( u \wedge v \) if and only if \( u \vee v \) covers \( u \) and \( v \), which is equivalent to modularity.

For the forward direction, we will show that the map \( \varphi \) from \( I = [u \wedge v, u] \) to \( J = [v, u \vee v] \) given by \( \varphi(x) = x \vee v \) is an isomorphism with inverse \( \psi: J \to I \) given by \( \psi(y) = y \wedge u \).

- To check that \( \varphi \) is well-defined, we need that for all \( x \in I \), \( \varphi(x) = x \vee v \geq v \). Since \( x \leq u \), \( u \vee v = (u \vee x) \vee v = u \vee (x \vee v) \geq x \vee v \). A similar argument shows that for \( y \in J \), \( \psi(y) \in I \).

- To see that \( \psi \circ \varphi \) is the identity on \( I \), for any \( x \in I \), let \( z = \psi(\varphi(x)) = (x \vee v) \wedge u \). Since \( x \leq u \) and \( x \leq x \vee v \), we have \( x \leq (x \vee v) \wedge u = z \). But by modularity, \( L \) is graded with rank function \( \rho \), and

\[
\begin{align*}
\rho(z) &= \rho((x \vee v) \wedge u) \\
&= \rho(u) + \rho(x \vee v) - \rho((x \vee v) \vee u) \\
&= \rho(u) + \rho(x \vee v) - \rho(u \vee v) \\
&= \rho(u) + (\rho(x) + \rho(v) - \rho(x \wedge v)) - (\rho(u) + \rho(v) - \rho(u \wedge v)) \\
&= \rho(x) - \rho(x \wedge v) + \rho(u \wedge v) \\
&= \rho(x),
\end{align*}
\]

\( \square \)
4. (a) Let $\alpha v$, $\sigma(c)$ Let $\bullet$

so we must have $z = x$. (Here we use that $(x \lor v) \lor u = (x \lor u) \lor v = u \lor v$, and $x \land v = (x \land u) \land v = x \land (u \land v) = u \land v$.)

A similar argument shows that $\varphi \circ \psi$ is the identity on $J$.

- To see that $\varphi$ is order-preserving, note that if $x \leq y$, then $y \lor v = (x \lor y) \lor v = y \lor (x \lor v) \geq x \lor v$. A similar argument shows that $\psi$ is also order-preserving.

(b) Let $\alpha_{ij}$, for $1 \leq i < j \leq n$, be the atom in $\Pi_n$ in which $i$ and $j$ lie in the same block and all other elements lie in blocks of size 1. Then $\Pi_n$ is atomic: any $\pi \in \Pi_n$ is the join, or common coarsening, of all $\alpha_{ij}$ for which $i$ and $j$ lie in the same block of $\pi$.

To see that $\Pi_n$ is semimodular, consider any set partition $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_k\} \in \Pi_n$. Then any $\tau$ that covers $\sigma$ is obtained by replacing some $\sigma_i$ and $\sigma_j$ with $\sigma_i \cup \sigma_j$. Now suppose $\tau_1$ and $\tau_2$ both cover $\sigma = \tau_1 \land \tau_2$. Assume without loss of generality that $\tau_1 = \{\sigma_1 \cup \sigma_2, \sigma_3, \ldots, \sigma_k\}$.

- If $\tau_2$ does not merge either $\sigma_1$ or $\sigma_2$, say $\tau_2 = \{\sigma_1, \sigma_2, \sigma_3 \cup \sigma_4, \ldots, \sigma_k\}$, then $\tau_1 \lor \tau_2 = \{\sigma_1 \cup \sigma_2, \sigma_3 \cup \sigma_4, \ldots, \sigma_k\}$, which covers both $\tau_1$ and $\tau_2$.

- If $\tau_2$ does merge one of $\sigma_1$ or $\sigma_2$, say $\tau_2 = \{\sigma_1 \cup \sigma_3, \sigma_2, \sigma_4, \ldots, \sigma_k\}$, then $\tau_1 \lor \tau_2 = \{\sigma_1 \cup \sigma_2 \cup \sigma_3, \sigma_4, \ldots, \sigma_k\}$, which again covers both $\tau_1$ and $\tau_2$.

Hence $\Pi_n$ is semimodular.

(b) Let $v_{ij} = e_i - e_j \in \mathbb{R}^n$, and let $S = \{e_i - e_j \mid 1 \leq i, j \leq n\}$. We claim that $L(S) \cong \Pi_n$.

For any subspace $W \subseteq V$, let $\sigma(W)$ be the set partition of $[n]$ for which $i$ and $j$ lie in the same block if and only if $e_i \equiv e_j$ in $V/W$, or equivalently, if and only if $v_{ij} \in W$. (Such a set partition exists because equality in $V/W$ is an equivalence relation.) This gives a bijection from subspaces $W$ spanned by subsets of $S$ to elements of $\Pi_n$: given a set partition $\sigma \in \Pi_n$, we can reconstruct $W$ by letting $W$ be the span of $e_i - e_j$, where $i$ and $j$ lie in the same block of $\sigma$. (This subspace $W$ is given by the linear equations $\sum_{i \in \sigma_k} x_i = 0$ for all blocks $\sigma_k \in \sigma$, from which it is clear that $e_i - e_j$ will lie in $W$ if and only if $i$ and $j$ lie in the same block $\sigma_k$.)

To see this is an isomorphism, note that $W_1 \cap S \subseteq W_2 \cap S$ if and only if $e_i - e_j \in W_1$ implies $e_i - e_j \in W_2$. This holds if and only if any two elements in the same block of $\sigma(W_1)$ also lie in the same block of $\sigma(W_2)$, or equivalently, $\sigma(W_1) \leq \sigma(W_2)$.

(c) Let $\sigma = 12|3456 \cdots n$, and let $\tau = 13|2456 \cdots n$. Then $\sigma \lor \tau$ is the common coarsening of $\sigma$ and $\tau$, which is $\hat{1} = 123456 \cdots n$, which covers both $\sigma$ and $\tau$. However, $\sigma \land \tau = 1|23|456 \cdots n$, which is not covered by $\sigma$ or $\tau$. Hence $\Pi_n$ is not modular.