Homework:

1. A plane partition \( \pi \) is a tableau of positive integers (of partition shape) such that each row and column is weakly decreasing. Let \( R(a, b, c) \) denote the number of plane partitions that fit in an \( a \times b \) rectangle and whose entries are all at most \( c \).

(a) Show that \( R(a, b, c) \) is symmetric in \( a, b, \) and \( c \).

(b) Show that

\[
R(a, b, c) = \frac{|a+b|}{a} \frac{|a+b|}{a+1} \ldots \frac{|a+b|}{a+c-1} \frac{|a|}{a-1} \frac{|a|}{a} \ldots \frac{|a|}{a+c-2} \ldots \frac{|b|}{b} \frac{|b|}{b+1} \ldots \frac{|b|}{b+c-1} \frac{|c|}{c-1} \frac{|c|}{c} \ldots \frac{|c|}{c+c-1} \frac{|c|}{c+c-2} \ldots \frac{|c|}{c+c}.
\]

(c) *A lozenge is a rhombus formed by joining two equilateral triangles of side length 1. Show that \( R(a, b, c) \) is the number of ways to tile an equiangular hexagon with side lengths \( a, b, c, a, b, c \) with lozenges.

(d) *Show that \( R(a, b, c) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2} \).

2. Let \( \delta = (n-1, n-2, \ldots, 1) \).

(a) Show that \( s_{\delta}(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i + x_j) \).

(b) Show that \( s_{\delta/(2)} = s_{\delta/(1,1)} \).

3. Show that \( \sum_{R} s_{R} = s_{1}^{n} \), where \( R \) ranges over all \( 2^{n-1} \) ribbons with \( n \) boxes (up to translation).

4. (a) Suppose \( \lambda \) does not contain a hook of size \( q \). Show that \( \lambda \) does not contain any hook whose size is a multiple of \( q \). (Such a partition is called a \( q \)-core.)

(b) Suppose \( \lambda \) is a \( q \)-core. Show that \( s_{\lambda} \) lies in the subalgebra of \( \Lambda \) generated by the power sums \( p_{i} \) for \( i \) not divisible by \( q \).

5. *Let \( s_{1}^{\perp} \) be the linear operator on \( \Lambda \) adjoint to multiplication by \( s_{1} \), i.e., \( s_{1}^{\perp} s_{\lambda} = s_{\lambda/(1)} \). Show that \( s_{1}^{\perp} f = \frac{\partial f}{\partial p_{1}} \) when \( f \in \lambda = C[p_{1}, p_{2}, \ldots] \) is written as a polynomial in the power sums.