1. (5 pts) There are three (common) 3-polytopes with exactly 6 vertices up to combinatorial equivalence. Sketch a Schlegel diagram for each one as well their polars.

**Solution:** The polytopes are: the pentagonal pyramid, which is self-dual; the triangular prism, which is dual to the triangular bipyramid; and the octahedron, which is dual to the cube.

(Actually there are six 3-polytopes with 6 vertices up to combinatorial equivalence; the others are less common.)
2. (4 pts) Let \( P \subseteq \mathbb{R}^n \) be a polyhedron, and suppose there exist \( y \in \mathbb{R}^n \) and \( x \in P \) such that \( x + ty \in P \) for all \( t \geq 0 \). Show that \( z + ty \in P \) for all \( z \in P \) and \( t \geq 0 \).

**Solution:** Suppose \( z + t_0 y \notin P \) for some \( t_0 > 0 \). By Farkas’ Lemma, there exists a linear functional \( a \in (\mathbb{R}^n)^* \) such that \( az' \leq a_0 \) for all \( z' \in P \), but

\[
a_0 < a(z + t_0 y) = az + t_0 ay.
\]

In particular, since \( z \in P \), we must have \( az \leq a_0 \), so \( ay > 0 \). But then

\[
a(x + ty) = ax + tay
\]

is unbounded above as \( t \to \infty \), which is a contradiction (since \( x + ty \in P \), it should be bounded by \( a_0 \)).

3. (4 pts) Show that if \( P \) is a \( d \)-polytope that is both simple and simplicial, then either \( d \leq 2 \) or \( P \) is a simplex.

**Solution:** Consider any vertex \( v \). Since \( P \) is simple, the vertex figure \( P/v \) is a \((d - 1)\)-simplex. Since \( P/v \) has exactly \( d \) vertices, any proper subset of which forms a face, it follows that \( v \) has exactly \( d \) neighboring vertices \( w_1, \ldots, w_d \), and any proper subset of these neighbors lies in a face with \( v \). In particular, if \( d > 2 \), then any two of the \( w_i \) must lie in a face together. But since \( P \) is simplicial, all of its faces are simplices, so any two vertices lying in a face together must be adjacent. Therefore each of \( v, w_1, \ldots, w_d \) is adjacent to all the others. Since each vertex has exactly \( d \) neighbors, there can be no other vertices, so \( P \) is a simplex.
4. Let \( P = \text{conv} \begin{bmatrix} 0 & -2 & 1 & 1 \\ 2 & -2 & 0 & 1 \end{bmatrix} \).

(a) (3 pts) Sketch \( P^\Delta \) and the normal fan of \( P \). (Be sure to label the vertices of \( P^\Delta \).)

**Solution:** We can compute the vertices of \( P^\Delta \) to be the points \( a \in (\mathbb{R}^2)^* \) such that the facets of \( P \) have the form \( ax \leq 1 \). The face fan of \( P^\Delta \) is then the normal fan of \( P \).

In the picture below, the face fan is given in blue, and \( P^\Delta \) is given in red.

(b) (3 pts) For any lattice point \( (x, y) \in P \cap \mathbb{Z}^2 \), let \( f(x, y) = x^2 + y^2 \). Draw the regular subdivision of \( P \) corresponding to the lower hull of the graph of \( f \).

**Solution:** Here, the numbers indicate the values of \( f \) at each point.

(c) (3 pts) How many points of the form \( (x, y) \) for \( x, y \in \mathbb{Z} \) lie in the interior of \( P \)?

**Solution:** Scaling up by 10, we need to count the number of lattice points in \( 10P \). Note that \( P \) has 5 lattice points on its boundary and 3 lattice points in its interior. By Pick’s theorem, it follows that \( \text{vol}(P) = 3 + \frac{5}{2} - 1 = \frac{9}{2} \). Thus \( 10P \) has volume \( 10 \cdot 10^2 \cdot \frac{9}{2} = 450 \) and has \( 10 \cdot 5 = 50 \) lattice points on its boundary. Then again by Pick’s theorem, it must have \( 450 - \frac{50}{2} + 1 = 426 \) interior lattice points.

Alternatively, the Ehrhart polynomial of \( P \) is \( i(k) = \frac{9}{2}k^2 + \frac{5}{2}k + 1 \). (For instance, the leading term comes from the volume of \( P \), the constant term is always 1, and the linear term can be determined from \( i(1) = 8 \).) Then the number of interior lattice points in \( 10P \) is \((-1)^2i(-10) = 426\), as desired.
5. Let \( x(t) = (t, t^2, t^3, t^4, t^5) \in \mathbb{R}^5 \), and let \( P = \text{conv}\{x(i) \mid i = 1, 2, 3, \ldots, 9\} \subseteq \mathbb{R}^5 \).

(a) (3 pts) Find the equation of a supporting hyperplane of \( P \) that defines an edge between \( x(2) \) and \( x(5) \).

**Solution:** We wish to find an inequality \( c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 \geq 0 \) that attains equality on \( x(2) \) and \( x(5) \) but that is strict at other points \( x(t) \). In other words, we want
\[
c(t) = (t - 2)^2(t - 5)^2 = (t^2 - 7t + 10)^2 = t^4 - 14t^3 + 69t^2 - 140t + 100,
\]
so we should take the hyperplane
\[
100 - 140x_1 + 69x_2 - 14x_3 + x_4 = 0.
\]

(b) (3 pts) What is the \( f \)-vector of \( P \)?

**Solution:** Clearly \( f_{-1} = 1 \) and \( f_0 = 9 \). Recall that \( P \) is a cyclic polytope, which is neighborly and simplicial. Hence \( f_1 = \binom{9}{2} = 36 \). The other face numbers can be determined from the Dehn-Sommerville equations. Using Stanley’s trick, we can find \( h_0, h_1, \) and \( h_2, \) and then use the symmetry of the \( h \)-numbers to find the remaining face numbers.

\[
\begin{array}{cccccc}
1 & 9 \\
1 & 8 & 36 \\
1 & 7 & 28 & 74 \\
1 & 6 & 21 & 46 & 75 \\
1 & 5 & 15 & 25 & 29 & 30 \\
1 & 4 & 10 & 10 & 4 & 1 \\
\end{array}
\]

Thus the \( f \)-vector is \((1, 9, 36, 74, 75, 30)\).

(One can also derive the first few \( h \)-numbers using the Upper Bound Theorem/g-theorem.)

(c) (2 pts) State the relevant case of the Upper Bound Theorem.

**Solution:** Since \( P \) is a cyclic 5-polytope with 9 vertices, the Upper Bound Theorem states that any 5-polytope with 9 vertices has at most as many faces of dimension \( i \) as \( P \) for all \( i \).

(a) (3 pts) Use the definition of shellability to explain why $\Delta$ is not shellable.

**Solution:**

Suppose there existed a shelling $\prec$ for $\Delta$. Let $F = 245$, $G_1 = 135$, and $G_2 = 356$. Note that these are the only facets that contain 5. Then we cannot have, for instance, $F \prec G_1, G_2$ since then whichever of $G_1$ or $G_2$ is added first will intersect the previous facets in the vertex 5 but not in any edge containing 5. Similarly, if $F \succ G_1$ or $F \succ G_2$, then $F$ will intersect the previous facets in the vertex 5 but not in any edge containing 5.

(b) (3 pts) Add a single facet to $\Delta$ to make a shellable simplicial complex $\Delta'$, and exhibit a shelling for $\Delta'$.

**Solution:** We can add the facet 235. (In fact, this is the only possible facet that can be added.) Then one possible shelling order is:


(c) (2 pts) What is the $h$-vector of $\Delta'$?

**Solution:** In the previous shelling order, the facets were attached along 0, 1, 1, 1, 2, 1, 2 of their edges, respectively. It follows that $h = (1, 4, 2, 0)$.

Alternatively, one can also compute the $h$-vector from the $f$-vector, $(1, 7, 13, 7)$. 

7. (4 pts) What are the possible $f$-vectors of a simplicial 4-polytope with 14 facets?

Solution: Suppose the polytope $P$ has $h$-vector $h = (h_0, h_1, h_2, h_3, h_4)$. By the Dehn-Sommerville equations, $h_0 = h_4 = 1$, and $h_1 = h_3$. Since $\sum_i h_i = 14$ is the number of facets of $P$, we must have that $h$ has the form

$$h = (1, h_1, 12 - 2h_1, h_1, 1).$$

In particular, $P$ has $g$-vector $g = (1, h_1 - 1, 12 - 3h_1)$. The $g$-theorem then states that

$$0 \leq 12 - 3h_1 \leq (h_1 - 1)^{(1)} = \binom{h_1}{2}.$$

A simple check shows that the only possibilities are $h_1 = 3$ or $h_1 = 4$. This gives $g$-vectors $(1, 2, 3)$ and $(1, 3, 0)$, which give $h$-vectors $(1, 3, 6, 3, 1)$ and $(1, 4, 4, 4, 1)$, which give $f$-vectors $(1, 7, 21, 28, 14)$ and $(1, 8, 22, 28, 14)$.

8. (a) (4 pts) Let $P$ be a lattice 3-polytope with $h^*$-polynomial $h(t) = 1 + h_1 t + h_2 t^2$. Express the $h^*$-polynomial of $2P$ in terms of $h_1$ and $h_2$.

Solution: The Ehrhart series of $P$ is

$$\text{Ehr}_{P}(t) = \frac{1 + h_1 t + h_2 t^2}{(1 - t)^4} = \sum_k i_P(k) t^k.$$

Since $i_{2P}(k) = i_P(2k)$, the Ehrhart series for $2P$ can be obtained from that of $P$ by extracting the coefficients with even exponents and substituting $t^2 \mapsto t$.

This can be done by computing

$$\frac{1}{2}(\text{Ehr}_{P}(t) + \text{Ehr}_{P}(-t)) = \frac{1}{2} \left( \frac{h(t)}{(1 - t)^4} + \frac{h(-t)}{(1 + t)^4} \right)$$

$$= \frac{1}{2} \left( h(t) \cdot (1 + t)^4 + h(-t) \cdot (1 - t)^4 \right).$$

Since the numerator is just the even degree terms of

$$h(t) \cdot (1 + t)^4 = h(t) \cdot (1 + 4t + 6t^2 + 4t^3 + t^4),$$

it follows that the $h^*$-polynomial of $2P$ is

$$1 + (h_2 + 4h_1 + 6)t + (6h_2 + 4h_1 + 1)t^2 + h_2 t^3.$$

Note: If $h(t)$ has a term $h_3 t^2$, the answer is only slightly more complicated,

$$1 + (h_2 + 4h_1 + 6)t + (4h_3 + 6h_2 + 4h_1 + 1)t^2 + (4h_3 + h_2)t^3.$$
(b) (4 pts) Find the Ehrhart polynomial and Ehrhart series of $Q = \text{conv} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$.

Solution: Since $Q = 2P$, where $P = \text{conv} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, we can use part (a) if we can find the Ehrhart series of $P$. Since it is a simplex with one vertex at the origin, this is straightforward; there is only one lattice point other than the origin in the fundamental domain $\Pi(V)$, namely $(1, 1, 1) = \frac{1}{2}(1, 1, 0) + \frac{1}{2}(1, 0, 1) + \frac{1}{2}(0, 1, 1)$. Since $\left\lceil \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right\rceil = 2$, this contributes to $h_2$. We therefore find that $P$ has $h^*_P(t) = 1 + t^2$. It follows from part (a) that $Q$ has $h^*$-polynomial $1 + 7t + 7t^2 + t^3$ and thus Ehrhart series

$$\frac{1 + 7t + 7t^2 + t^3}{(1-t)^4}$$

and Ehrhart polynomial

$$\binom{k+3}{3} + 7\binom{k+2}{3} + 7\binom{k+1}{3} + \binom{k}{3} = 1 + \frac{10}{3}k + 4k^2 + \frac{8}{3}k^3.$$  

Alternatively, one can compute the Ehrhart polynomial of $Q$ using evaluations such as

$$i_Q(0) = 1$$
$$i_Q(1) = |Q \cap Z^3| = 11$$
$$i_Q(-1) = (-1)^3 \cdot |Q^c \cap Z^3| = -1$$

together with the fact that the leading coefficient is given by the volume of $Q$. 