1. The *Minkowski sum* of two sets $S, T \subseteq \mathbb{R}^d$ is defined by

$$S + T = \{x + y \mid x \in S, y \in T\}.$$ 

Show that the Minkowski sum of two V-polytopes is again a V-polytope.

2. Let $S$ be a set of $d + 2$ points in $\mathbb{R}^d$. Show that $S$ can be partitioned into two disjoint (nonempty) sets $A$ and $B$ such that $\text{conv}(A) \cap \text{conv}(B) \neq \emptyset$.

3. Consider the $d$-dimensional hypercube $C_d = \{(x_1, \ldots, x_d) \mid -1 \leq x_i \leq 1\}$.

   (a) Given $a \in (\mathbb{R}^d)^*$, for which point(s) $x \in C_d$ is $ax$ maximized?

   (b) Show that $C_d$ has $3^d$ nonempty faces. How many faces of dimension $k$ does it have for each $0 \leq k \leq d$?

4. An $n \times n$ matrix with nonnegative real entries is called *doubly stochastic* if the sum of each row and column is 1. The set of all $n \times n$ doubly stochastic matrices in $\mathbb{R}^{n^2}$ is the *Birkhoff polytope* $P(n)$. Show that $P(n)$ is the convex hull of the $n!$ permutation matrices.