1. (a) Let \( \prec \) denote colexicographic order on the \( k \)-faces of a simplex. If \( G \prec F \), then the largest element \( x \) in which they differ lies in \( F \). Choose any \( y \in G \setminus F \), so \( x > y \). Then \( H = (F \setminus \{x\}) \cup \{y\} \prec F \). Moreover, \( G \cap F \subseteq F \setminus \{x\} = H \cap F \), and \( H \cap F \) is a facet of \( F \). It follows that \( \prec \) is a shelling order.

(b) Let \( \Delta \) be a shellable simplicial complex of dimension \( d \). It suffices to check that each time we attach a new facet in the shelling of \( \Delta \), we can extend the shelling to contain the new \( k \)-faces. The case \( k = d \) is trivial, so assume \( k < d \).

Suppose that when we attach the facet \( F \) to the partial complex \( C \subseteq \Delta \), the minimal new face added is \( R \subseteq F \). Order the vertices of \( F \) so that all elements of \( R \) come after all elements not in \( R \). Then the colexicographic order on the \( k \)-skeleton of \( F \), which is a shelling by part (a), has all the new \( k \)-faces \( H_1, H_2, \ldots \) occurring after all the \( k \)-faces of \( F \) already in \( C \).

We claim that adding the new \( k \)-faces \( H_1, H_2, \ldots \) according to this shelling order extends the shelling of the \( k \)-skeleton of \( C \) to the \( k \)-skeleton of \( C \cup F \). Indeed, consider any new \( k \)-face \( H_i \) and a previously added \( k \)-face \( G \). If \( G \neq H_j \) for any \( j \), then \( G \) is a \( k \)-face of \( C \) and therefore does not contain \( R \). Hence \( G \cap H_i \) is contained in a facet of \( H_i \) that does not contain \( R \), which therefore already lies in \( C \). If instead \( G \) is some \( H_j \), then we know that \( H_i \cap H_j \) is contained in some facet of \( H_i \) that has already been added due to the shelling order of \( F \). The result follows.

2. (a) Yes. By the Kruskal-Katona theorem, we verify that

\[
27 \leq 8^{(1)} = \binom{8}{2} = 28,
\]

\[
36 \leq 27^{(2)} = \binom{7}{3} + \binom{6}{2} = 50,
\]

\[
18 \leq 36^{(3)} = \binom{7}{4} + \binom{6}{3} + \binom{0}{2} = 35.
\]

(b) Yes. We must verify that the corresponding \( h \)-vector is an \( M \)-sequence. First, using Stanley’s trick:

\[
\begin{array}{cccc}
1 & 8 & 7 & 27 \\
1 & 6 & 20 & 36 \\
1 & 5 & 14 & 16 & 18 \\
1 & 4 & 9 & 2 & 2
\end{array}
\]

Thus \( h = (1, 4, 9, 2, 2) \). Then we verify that

\[
9 \leq 4^{(1)} = \binom{5}{2} = 10,
\]
\[2 \leq 9^{(2)} = \binom{5}{3} + \binom{4}{2} = 16,\]
\[2 \leq 2^{(3)} = \binom{4}{4} + \binom{3}{3} + \binom{1}{2} = 2.\]

(c) No. The \(h\)-vector found in part (b) is not symmetric, so it does not satisfy the Dehn-Sommerville equations.

3. (a) Note that any simplicial \(d\)-polytope with \(n\) vertices has \(f\)-vector of the form \((1, n, \ldots)\) and hence \(h\)-vector of the form \((1, n-d, \ldots)\) and \(g\)-vector of the form \((1, n-d-1, \ldots)\).

Since the \(f\)-numbers are positive linear combinations of the \(g\)-numbers, it suffices to show that the \(g\)-numbers for \(S_d(n)\) are as small as possible, that is, \(g = (1, n - d - 1, 0, 0, 0, \ldots)\). Equivalently, we must have \(h = (1, n - d, n - d, \ldots, n - d, 1)\).

We induct on \(n\). If \(n = d + 1\), then \(S_d(\cdot + 1)\) is a \(d\)-simplex, which has \(h\)-vector \((1, 1, \ldots, 1)\). Suppose we pass from \(S_d(n)\) to \(S_d(n + 1)\) by adding the vertex \(x\) beyond the facet \(F\). Then we effectively remove the facet \(F\) but add the faces \(\text{conv}(\{x\} \cup G)\) for all proper faces \(G \subseteq F\). This changes the \(f\)-polynomial by

\[x^{d-1} + dx^{d-2} + \binom{d}{2} x^{d-3} + \cdots + \binom{d}{d-2} x + \binom{d}{d-1} - 1 = \frac{(1 + x)^d - 1}{x} - 1.\]

Hence the \(h\)-polynomial gets changed by \(\frac{x^{d-1}}{x-1} - 1 = x^{d-1} + x^{d-2} + \cdots + x\), that is, \(h_i\) gets increased by 1 for \(1 \leq i \leq d - 1\). The result follows easily.

(Alternatively, one can use a shelling argument: take a shelling of \(S_d(n)\) that adds \(F\) last, such as a line shelling through \(x\), but instead of adding \(F\), add the new facets of \(S_{d+1}(n)\) containing \(x\). This reduces \(h_d\) by 1 but then increases \(h_1, h_2, \ldots, h_d\) each by 1.)

(b) No. For instance, \(S_3(5)\) has 6 facets, but the square pyramid has only 5 facets.