1. Let $b_j, i_j$ denote the number of boundary and interior points of $P_j$. Then

$$\text{vol } P = \text{vol } P_1 - \text{vol } P_2 = i_1 + \frac{b_1}{2} - 1 - (i_2 + \frac{b_2}{2} - 1) = i_1 - (i_2 + b_2) + \frac{b_1 + b_2}{2}.$$  

Now note that $i = i_1 - (i_2 + b_2)$ is the number of interior points of $P$, while $b = b_1 + b_2$ is the number of boundary points of $P$. Hence $\text{vol } P = i + \frac{b}{2}$ for polygons with a hole.

2. Let $T_r \subseteq \mathbb{R}^3$ be the tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0)$ and $(1,1,r)$. The volume of $T_r$ is

$$\frac{1}{3!} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & r \end{vmatrix} = \frac{r}{6}.$$ 

We will show that $T_r$ contains no lattice points other than its vertices. Any point of $T_r$ has the form

$$\lambda_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \lambda_4 \begin{bmatrix} 1 \\ 1 \\ r \end{bmatrix} = \lambda_2 + \lambda_4 \begin{bmatrix} \lambda_2 + \lambda_4 \\ \lambda_3 + \lambda_4 \\ r \lambda_4 \end{bmatrix},$$

where $\sum \lambda_i = 1$ and $0 \leq \lambda_i \leq 1$. For this to be a lattice point, we must have $\lambda_4 = \frac{k}{r}$ for some integer $0 \leq k \leq r$. The cases $k = 0$ or $r$ give the four vertices of $T_r$. If $0 < k < r$, then we must have $\lambda_2 = \lambda_3 = 1 - \frac{k}{r}$ to get an integer point, but then $\sum \lambda_i > 1$, which is a contradiction.

3. Let $P$ be a poset on $[n]$. Clearly $O(P)$ lies inside the unit hypercube. Recall that the hyperplanes $x_i = x_j$ split the unit hypercube into $n!$ simplices

$$\Delta_\sigma = \{ \mathbf{x} \mid 0 \leq x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)} \leq 1 \}$$

for any permutation $\sigma \in S_n$, each of volume $\frac{1}{n!}$. A point in the interior of $\Delta_\sigma$ lies in $O(P)$ if and only if $\sigma^{-1}(s) < \sigma^{-1}(t)$ whenever $s < t$ in $P$, that is, if and only if $\sigma^{-1}$ is a linear extension of $P$. It follows that $O(P)$ is a union of $e(P)$ of the $\Delta_\sigma$, so its volume is $\frac{e(P)}{n!}$.

4. (a) Any lattice polygon has a unimodular triangulation, so it suffices to prove the result when $P$ is a unimodular triangle. Note that $P$ has at least 6 “half-lattice” points, namely the three vertices and the midpoints of the three sides (which are all midpoints of two lattice points in $P$). Hence $2P$ has at least 6 lattice points. Since $\text{vol}(2P) = 2^2 \text{vol}(P) = 2$, Pick’s theorem states that

$$2 = \text{vol}(2P) = i + \frac{b}{2} - 1 \geq 0 + \frac{6}{2} - 1 = 2.$$ 

Since equality holds, $2P$ cannot have any additional lattice points, so $P$ cannot have any additional half-lattice points.

(Alternatively, there is an affine transformation that preserves the lattice $\mathbb{Z}^2$ and sends the unimodular simplex $P$ to the unimodular simplex $\Delta = \text{conv}(0, e_1, e_2)$, for if $P = \text{conv}(u, u + v_1, u + v_2)$, then $v_1$ and $v_2$ form a basis for $\mathbb{Z}^2$. Since this map also preserves the “half-lattice,” it suffices to prove the result for $\Delta$, which is easy.)
(b) Take, for instance, the polytope $T_r \subseteq \mathbb{R}^3$ constructed in #2 above with $r > 1$. The point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ lies in $T_r$: take $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\frac{1}{2r}, \frac{1}{2r}, \frac{1}{2r}, \frac{1}{2r})$. However, it is evidently not the midpoint of any two lattice points in $T_r$. 