1. The simplicial cone $C = \text{cone}(V)$ is isomorphic to the positive orthant $\text{cone}(e_1, \ldots, e_d)$ using the linear transformation $V^{-1}$. Since the positive orthant is also $P(-I_d, 0)$, it follows that $C = \text{cone}(V) = P(-I_d \cdot V^{-1}, 0) = P(-V^{-1}, 0)$. Hence $A = -V^{-1}$, and the rows of $A$ are linearly independent. The converse direction is similar.

2. Consider any polytope $P = P(A, z) \subseteq \mathbb{R}^d$. Since $P$ is bounded, the rank of $A$ must be $d$ (since $P$ cannot contain a line and $P$ contains the kernel of $A$). Thus the columns of $A$ are linearly independent, so we can add columns to $A$ to form an invertible $d \times d$ matrix $A'$. Then $P$ is the intersection of $P' = P(A', (z, 0))$ with the linear subspace $\{(x, t) \mid t = 0\}$. Since $A'$ is invertible, $P'$ is affinely isomorphic to the positive orthant. But any bounded intersection of the positive orthant with an affine subspace is also the intersection of a simplex with an affine subspace: since the intersection is bounded, we can first intersect the positive orthant with the halfspace $\sum x_i \leq M$ for $M$ sufficiently large (which forms a simplex) and then intersect with the affine subspace.

3. We first homogenize the equations by introducing a variable $x_0$. We find that $C(P)$ is $P(A', 0)$, where

$$A' = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix},$$

so $P = C(P) \cap \{x \mid x_0 = 1\}$. (Actually, the first row is redundant since it is the sum of the second, third, and fifth row, but we will leave it in for now.)

Since the first four rows of $A'$ are clearly linearly independent, we can form an invertible matrix by adding on additional columns corresponding to $x_4$, $x_5$, and $x_6$.

$$A'' = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Then $C(P) = P(A'', 0) \cap \{x \mid x_4 = x_5 = x_6 = 0\}$.
Since $A''$ is invertible, by problem #1 above, we can write $C(P) = \text{cone}(V)$, where

$$V = -A''^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & -1 & -1 & 0 & -1 & 0 & 0 \\
1 & -1 & 0 & -1 & 0 & -1 & 0 \\
1 & 0 & -1 & -1 & 0 & 0 & -1
\end{bmatrix}$$

(Note that in this case, this is especially easy to compute because $A''$ has the form $\begin{bmatrix} -I & 0 \\ B & I \end{bmatrix}$, whose negative inverse is $\begin{bmatrix} I & 0 \\ -B & -I \end{bmatrix}$.)

Now we use Fourier-Motzkin elimination to intersect with the hyperplane where $x_6 = 0$: we keep any column with last coordinate 0 and combine any columns with last coordinates of opposite sign. This gives:

$$V' = \begin{bmatrix} v_2 & v_1 + v_3 & v_1 + v_4 & v_5 & v_6 & v_1 + v_7 \end{bmatrix} = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 1 & -1 & 0 & 1 \\
-1 & 1 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$  

We repeat to set $x_5 = 0$:

$$V'' = \begin{bmatrix} v_1' & v_2' & v_4' & v_1' + v_2' & v_1 + v_4 & v_5 & v_1 + v_4 & v_6 & v_1 + v_7 & v_5 + v_6 \end{bmatrix} = \begin{bmatrix}
1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$  

Finally, we repeat one more time to set $x_4 = 0$:

$$V''' = \begin{bmatrix} v_1'' & v_2'' & v_3'' & v_4' + v_2' & v_1' + v_3' & v_2' + v_5' & v_3' + v_5' + v_1 + v_7 & v_5 + v_6 \end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 & 2 & 1 & 2 \\
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$  

Removing the last three rows and scaling the columns to have first coordinate 1 gives that

$$C(P) = \text{cone} \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}, \quad \text{so} \quad P = \text{conv} \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}.$$
4. (a) Let $a \in (\mathbb{R}^*)^d$. We first show that, among all permutations $\pi = (\pi_1, \ldots, \pi_d)^T$, $a\pi = \sum a_i\pi_i$ attains its maximum value if $\pi_i > \pi_j$ whenever $a_i > a_j$. Indeed, suppose $a_i > a_j$ but $\pi_i < \pi_j$. Then if $\pi' = \pi_i$ and $\pi'_i = \pi_j$ and keeping all other entries the same, we find that

$$a(\pi' - \pi) = a_i\pi_j + a_j\pi_i - a_i\pi_i - a_j\pi_j = (a_i - a_j)(\pi_j - \pi_i) > 0,$$

so $a\pi$ cannot be maximized (since $a\pi < a\pi'$).

It follows that to maximize $a\pi$, we must have that $\pi_i < \pi_j$ implies $a_i \leq a_j$. In other words, $a$ determines an ordered set partition of $[d]$, that is, a partition of $[d]$ into sets $\sigma_1, \ldots, \sigma_m \subseteq [d]$, where $a_i \leq a_j$ if and only if $i \in \sigma_p$ and $j \in \sigma_q$ with $p \leq q$. (Hence $m$ is the number of distinct coordinates appearing in $a$.) Then $\pi$ maximizes $a\pi$ if $\pi$ sends elements of $\sigma_1$ to the smallest possible values $1, 2, \ldots, |\sigma_1|$ in some order, then elements of $\sigma_2$ to the next smallest values and so forth. Denote by $S(a)$ this set of permutations. Clearly any $\pi \in S(a)$ will attain the same maximum value $a\pi$.

Then any $x = \sum \mu_{\pi}\pi \in \Pi_{d-1}$ will attain this maximum value provided that $\mu_{\pi}$ is only nonzero for $\pi \in S(a)$, that is, provided $x \in \text{conv}_{\pi \in S(a)}(\pi)$. Any permutation in $S(a)$ can be obtained from any other by permuting the coordinates in each of the $\sigma_i$. We may apply a translation to $S(a)$ to shift the values of $\pi$ on the elements of a block $\sigma_i$ to be $1, \ldots, |\sigma_i|$. It is then immediate that $\text{conv}_{\pi \in S(a)}(\pi)$ is affinely isomorphic to $\Pi_{\lambda_1-1} \times \cdots \times \Pi_{\lambda_m-1}$, where $\lambda_i = |\sigma_i|$.

(b) From the argument above, since $\dim \Pi_{\lambda_1-1} \times \cdots \times \Pi_{\lambda_m-1} = d - m$, the number of faces of dimension $k$ is equal to the number of ordered set partitions of $[d]$ into $d - k$ sets. This is $(d - k)! \cdot S(d, d - k)$, where $S(d, d - k)$ is a Stirling number of the second kind.