1. (a) Let $A \in \mathbb{R}^{m \times d}$, $z \in \mathbb{R}^m$, and $b \in (\mathbb{R}^d)^*$. Suppose the polyhedron $P(A,z)$ is nonempty, and suppose $bx$ for $x \in P(A,z)$ attains a (finite) maximum value $M$. Show that $M$ is equal to the minimum value $m$ attained by $cz$, where $cA = b$ and $c \geq 0$.

(b) What happens if $\max_{x \in P(A,z)} bx = \infty$?

2. Let $P, Q \subseteq \mathbb{R}^d$ be disjoint convex polytopes. Show that there exists an affine hyperplane $H \subseteq \mathbb{R}^d$ such that $P$ and $Q$ lie (strictly) on opposite sides of $H$.

3. Show that every nonempty face of a polyhedral cone contains its lineality space.

4. Let $S$ be a finite set of open half-spaces of $\mathbb{R}^d$ such that $\bigcup_{H \in S} H = \mathbb{R}^d$. Show that there exists a subset $T \subseteq S$ of size at most $d + 1$ such that $\bigcup_{H \in T} H = \mathbb{R}^d$. 