1. (a) Let \( m = \min cz \). Since the inequality \( bx \leq M \) is valid on \( P(A, z) \), by Farkas’ Lemma, there exists \( c \in (R^m)^* \) such that \( c \geq 0 \), \( cA = b \) and \( cz \leq M \). Thus \( m \leq M \). On the other hand, if \( cz = m \), then for any \( x \in P(A, z) \), \( bx = cAx \leq cz = m \), so \( M \leq m \). Hence \( M = m \), as desired.

(b) If \( M = \infty \), then we claim that there is no \( c \geq 0 \) such that \( cA = b \). Indeed, if there were, then the argument above would show that, for all \( x \in P(A, z) \), \( bx \leq cz < \infty \). But \( bx \) is unbounded, so this is impossible.

2. Let

\[
P - Q = P + (-Q) = \{ p - q \mid p \in P, q \in Q \} = \text{conv}\{ v_i - w_j \mid v_i \in V, w_j \in W \},
\]

where \( P = \text{conv}(V) \) and \( Q = \text{conv}(W) \). (See HW 1, #1.) Since \( P \) and \( Q \) are disjoint, \( 0 \notin P \cap Q \). Hence by Farkas’ Lemma, there exists a linear functional \( a \in (R^d)^* \) such that \( ax \geq a_0 \) for all \( x \in P - Q \) and \( 0 = a0 < a_0 \). Thus \( a(p - q) = ap - aq \geq a_0 > 0 \) for all \( p \in P, q \in Q \). Hence there exists a real number \( b \) between \( \inf_{p \in P} ap \) and \( \sup_{q \in Q} aq \), and then the hyperplane \( ax = b \) separates \( P \) and \( Q \).

3. Suppose the face \( F \) of the cone \( C \) is defined by the valid inequality \( ax \leq a_0 \). Since \( 0 \in C \), we must have \( a_0 \geq 0 \). Suppose \( a_0 > 0 \). Choose any \( y \in F \). Then \( ay = a_0 \), but \( ty \in C \) for all \( t \geq 0 \), so \( ta_0 = a(ty) \leq a_0 \) for all \( t \geq 0 \). This is only possible if \( a_0 = 0 \), so \( 0 \in F \). Now if \( y \) lies in the lineality space of \( C \), so does \( -y \), so \( ay \leq 0 \) and \( -ay \leq 0 \) implies \( ay = 0 \), so \( y \) lies in \( F \) as well.

4. Consider any collection of half-spaces given by the rows of the matrix equation \( Ax > z \). These half-spaces cover all of \( R^d \) if and only if there is no solution to \( Ax \leq z \). By Farkas’ Lemma (I), this occurs if and only if there exists \( c \in (R^d)^* \) such that \( c \geq 0 \), \( cA = 0 \), and \( cz = -1 \), or equivalently, the vector \( (0, -1) \) lies in the cone generated by \( (a_i, z_i) \) (where \( a_i \) are the rows of \( A \), and \( z_i \) are the entries of \( z \)).

Now take \( S \) such that \( \bigcup_{H \in S} H = R^d \). Then \( (0, -1) \) lies in the cone generated by the \( (a_i, z_i) \), so by Carathéodory’s Theorem (for cones), it follows that \( (0, -1) \) lies in the cone generated by at most \( d + 1 \) of them. The corresponding half-spaces of \( S \) then cover \( R^d \).