MA 724 Homework 5 – Solutions

1. (a) If $Q$ is a pyramid with vertex $v$, then $P$ is the vertex figure $Q/v$. Hence the faces of $P$ are in bijection with the faces of $Q$ containing $v$ simply by adding the vertex $v$. The only other faces of $Q$ are those contained in $P$ itself, so it follows that $L(Q) \cong L(P) \times 2$.

(b) Let $v_1$ and $v_2$ be the two vertices of the bipyramid. Then, as above, the faces of $P$ are in bijection with the faces of $Q$ containing $v_i$. However, the face $Q$ contains both $v_1$ and $v_2$, and $P$ is no longer a face of $Q$. In summary, each face of $Q$ is either $Q$ itself or has the form $F$, $\text{conv}(F,v_1)$, or $\text{conv}(F,v_2)$ for some proper face $F$ of $P$.

In other words, let $L(P) = L'(P) \oplus 1$, so $L'(P)$ is the poset of proper faces of $P$. Then $L(Q) \cong (L'(P) \times V) \oplus 1$, where $V = 1 \oplus (1 + 1)$.

(c) Suppose $Q = P \times [0,1] \subseteq \mathbb{R}^{d+1}$. Let $a \in (\mathbb{R}^d)^*$ be maximized on the nonempty face $F$ of $P$. If $a' = [a/c] \in (\mathbb{R}^{d+1})^*$, then it is maximized on $Q$ at $F \times \{1\}$ if $c > 0$; on $F \times \{0\}$ if $c < 0$; and on $F \times [0,1]$ if $c = 0$. Adding in the empty face, we find that if $L(P) = 1 \oplus \Lambda(P)$, so $L'(P)$ is the poset of nonempty faces of $P$, then $L(Q) \cong 1 \oplus (\Lambda(P) \times \Lambda)$, where $\Lambda = (1 + 1) \oplus 1$.

(Alternatively, one can check that replacing $P$ with its prism is equivalent to replacing $P^\Delta$ with its bipyramid.)

2. (a) Clearly if $cx \leq 0$ for all $x \in C$, then $cx \leq 1$ for all $x \in C$. Conversely, suppose $cx \leq 1$ for all $x \in C$. Since $C$ is closed under taking positive scalar multiples, we must have $c(tx) = tcx \leq 1$ for all $t > 0$. Hence $cx \leq \frac{1}{t}$ for all $t > 0$, so $cx \leq 0$.

(b) We claim that $C^\Delta$ has H-representation given by $P(Y^T,0) = \{c \mid cY \leq 0\}$ and V-representation given by $\text{cone}(A^T) = \text{cone}\{a_i \mid a_i \in A\}$.

For the first claim, clearly if $c \in C^\Delta$, then $cy_i \leq 0$ for all $y_i \in Y$. Conversely, if $cy_i \leq 0$ for all $y_i \in Y$, then $c(\sum \lambda_i y_i) = \sum \lambda_i (cy_i) \leq 0$ for all $\lambda_i \geq 0$, so $c \in C^\Delta$.

For the second claim, if $c = \sum \lambda_i a_i \in \text{cone}(A^T)$, then $cx = \sum \lambda_i a_i x \leq 0$ for all $x \in C$. Conversely, suppose $cx \leq 0$ for all $x \in C = P(A,0)$. Then by Farkas’ Lemma, $cx \leq 0$ can be written as a linear combination of the inequalities $a_i x \leq 0$ defining $C$; hence $c \in \text{cone}(A^T)$.

Now $C$ is pointed $\iff$ its lineality space $\ker(A)$ is trivial $\iff A$ has rank $d$ $\iff A^T$ has rank $d$ $\iff$ the row span of $A^T$ is $\mathbb{R}^d$ $\iff C^\Delta = \text{cone}(A^T)$ has dimension $d$.

Similarly, $C = \text{cone}(Y)$ has dimension $d$ $\iff Y$ and $Y^T$ have rank $d$ $\iff C^\Delta = P(Y^T,0)$ is pointed.

(c) If $C$ is the homogenization of a polytope $P$, then $C = \text{cone}\{(1,x) \mid x \in P\}$. Then $(-1,c) \in C^\Delta$ if and only if $-1 + cx \leq 0$, that is, $cx \leq 1$. Hence $C^\Delta$ intersected with the plane $x_0 = -1$ is $P^\Delta$ (so $C^\Delta$ is a reflection of the homogenization of $P^\Delta$).
3. (a) Suppose $P$ is reflexive and $v ∈ \text{int}(P)$ is a nonzero lattice point. Then some point $tv$ for $t > 1$ lies on a facet whose affine span is $av = 1$, where $a$ is a vertex of $P^\Delta$ and hence a lattice point in $(\mathbb{R}^d)^*$. But then $av = \frac{1}{t}$ is not an integer, which is a contradiction.

(b) Let $P \subseteq \mathbb{R}^3$ be given by the inequalities $x_i ≤ 1$, $-x_i ≤ 1$, and $x_1 + x_2 + x_3 ≤ 2$. (In other words, $P$ is the convex hull of all lattice points with coordinates 0, 1, or $-1$ except for $(1, 1, 1)$.) Then $P$ contains no lattice points in its interior other than the origin (since the only way for an integer to satisfy $x_i < 1$ and $-x_i < 1$ is if $x_i = 0$). But $P^\Delta$ has the vertex $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, so $P$ is not reflexive.