1. Let $P$ and $Q$ be polytopes in $\mathbb{R}^d$. Show that each face of the Minkowski sum $P + Q$ has the form $F + G$, where $F$ is a face of $P$ and $G$ is a face of $Q$.

Deduce that the normal fan $N(P + Q)$ is the common refinement of $N(P)$ and $N(Q)$.

2. Show that the following diagram is not Schlegel.

3. Let $S = \{v_1, \ldots, v_n\}$ be the vertices of a convex set whose affine span is $\mathbb{R}^d$. For any function $f: S \to \mathbb{R}$, let $\Delta_f$ be the regular subdivision of $\text{conv}(S)$ obtained from the lower hull of $f$.

For any regular subdivision $\Delta$ of $\text{conv}(S)$ whose vertices lie in $S$, define $C(S, \Delta)$ to be the set of functions $f \in \mathbb{R}^S$ such that $\Delta_f$ is a coarsening of $\Delta$—i.e., each face of $\Delta$ is contained in a face of $\Delta_f$.

(a) Show that $C(S, \Delta)$ is a polyhedral cone in $\mathbb{R}^S$. Is it pointed?

(b) Show that if $\Delta'$ is a coarsening of $\Delta$, then $C(S, \Delta')$ is a face of $C(S, \Delta)$.

(c) Show that the set of $C(S, \Delta)$ forms a complete fan $\Sigma_S$ in $\mathbb{R}^S$, called the secondary fan of $S$.

(d) Let $S = \{(0, 0), (0, 1), (1, 0), (1, 2), (2, 1)\}$. Draw the restriction of $\Sigma_S$ to the subspace 
$$\{f \in \mathbb{R}^S \mid f(0, 0) = f(0, 1) = f(1, 0) = 0\},$$
and identify the subdivision $\Delta$ corresponding to each cone in $\Sigma_S$. 