About homework: Homework is due at the beginning of class on the date indicated. Collaboration is encouraged, but you are expected to write up your own solutions.
*Starred problems are optional and will not be graded.

1. Call a subset of \( n + 2 \) points in \( \mathbb{P}^n \) in general position if no \( n + 1 \) of them lie on a hyperplane. Show that given any two such subsets, there exists a projective linear transformation taking one to the other.

2. For any vector space \( V \) of dimension \( n \), let \( G(k,V) \) denote the Grassmannian of \( k \)-dimensional subspaces of \( V \). Show that there exists a canonical bijection between \( G(k,V) \) and \( G(n-k,V^*) \).

3. (a) Show that the map \( \sigma: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3 \) given by

\[
\sigma([X_0 : X_1], [Y_0 : Y_1]) = [X_0 Y_0 : X_0 Y_1 : X_1 Y_0 : X_1 Y_1]
\]

is a (well-defined) injective map onto the projective variety \( Q \subseteq \mathbb{P}^3 \) defined by the equation \( Z_0 Z_3 - Z_1 Z_2 = 0 \).

(b) Show that for any \( p \in \mathbb{P}^1 \), \( \sigma(\{p\} \times \mathbb{P}^1) \) and \( \sigma(\mathbb{P}^1 \times \{p\}) \) are both lines, and moreover show that all lines contained in \( Q \) have one of these two forms.

(c) *Show that any three skew lines in \( \mathbb{P}^3 \) can be sent via a projective linear transformation to the lines \( \sigma(\{[1 : 0]\} \times \mathbb{P}^1) \), \( \sigma(\{[0 : 1]\} \times \mathbb{P}^1) \), and \( \sigma(\{[1 : 1]\} \times \mathbb{P}^1) \).

(d) *Show that a generic line in \( \mathbb{P}^3 \) intersects \( Q \) in exactly two points, and if a line intersects \( Q \) in at least three distinct points, then it must lie entirely in \( Q \).

(e) *Conclude that, given four generic lines in \( \mathbb{P}^3 \), there are two lines that intersect all four.