1. Recall that $v \leq w$ if and only if $r_v(p,q) \geq r_w(p,q)$, where $r_v(p,q)$ is the number of 1’s in the upper left $p \times q$ submatrix of $v$. Note that $r_v(p,q)$ is the number of $v_1, \ldots, v_q$ that are at most $p$. In other words, it is the number of entries in column $n - q$ of $K(v)$ that are at most $p$. Hence each column of $K(v)$ must have at least as many entries that are at most $p$ as the corresponding column of $K(w)$. In particular, if we set $p$ to entry $(i, j)$ in $K(w)$, we must have that column $j$ of $K(v)$ has at least $i$ entries at most $p$, so the $(i, j)$ entry of $K(v)$ is at most $p$. Thus $K(v) \leq K(w)$ entrywise. The reverse direction is similar.

2. We induct on $n$. The first row of any reduced pipe dream must have at least $c_1 = w_1 - 1$ crosses so that the pipe $w_1$ can exit the first row. If there are no other crosses in that row, then the remainder of the pipe dream must represent the permutation $w'$ that sends $12\cdots w_1 \cdots n$ to $w_2 w_3 \cdots w_n$. The code of $w'$ is then $(c_2, c_3, \ldots, c_n)$ (since $c_i$ is the number of letters after $w_i$ that are smaller than it, which is unchanged when we remove $w_1$). Hence by induction, the smallest way to fill the rows past the first lexicographically has weight $x_2^{c_2} x_3^{c_3} \cdots$. Multiplying by $x_1^{c_1}$ for the first row gives the result.

3. We claim that if $u$ is any permutation such that $1 \leq u_i \leq n$ for all $1 \leq i \leq n$, then in any reduced pipe dream for $u$, the first $n$ pipes stay within the staircase shape $\delta$ of size $n$. Indeed, if not, then some wire $m$ with $m > n$ must enter $\delta$ by crossing some wire $i \leq n$. But $m$ does not form an inversion with $i$, so this is impossible.

Any such $u$ can be written as $u = vw$, where $v \in S_n$ and $w \in S_\infty$ with $w_i = i$ for $1 \leq i \leq n$. Then in any reduced pipe dream for $u$, $\delta$ contains a reduced pipe dream for $v$, and the remainder contains a pipe dream for $w$ if we place elbows in all squares in $\delta$. Since, by above, any reduced pipe dream for $w$ must contain elbows in all squares of $\delta$, this gives a bijection between pipe dreams for $u$ and pairs of pipe dreams for $v$ and $w$ that implies $S_u = S_v S_w$.

4. (a) Write

$$c_1 x_1 + \cdots + c_n x_n = \sum_{i=1}^{n} (c_i - c_{i+1})(x_1 + \cdots + x_i) = \sum_{i=1}^{n} (c_i - c_{i+1}) S_i$$

(where we set $c_{n+1} = 0$). Then by Monk’s rule,

$$(c_1 x_1 + \cdots + c_n x_n) S_w = \sum_{i=1}^{n} \sum_{\ell(w_{tj}) = \ell(w) + 1}^{k-1} (c_i - c_{i+1}) S_{wt_{jk}}$$

$$= \sum_{\ell(w_{tj}) = \ell(w) + 1}^{k-1} \sum_{i=j}^{k-1} (c_i - c_{i+1}) S_{wt_{jk}}$$

$$= \sum_{\ell(w_{tj}) = \ell(w) + 1} (c_j - c_k) S_{wt_{jk}}.$$
(b) Let \( v = wt_{pq} \). Then by part (a),

\[
x_p \mathcal{G}_v = - \sum_{r<p, \ell(vt_{rp}) = \ell(v)+1} \mathcal{G}_{vt_{rp}} + \sum_{s>p, \ell(vt_{ps}) = \ell(v)+1} \mathcal{G}_{vt_{ps}}.
\]

We claim that there is a unique choice of \( s \) for the second sum, namely \( s = q \). Indeed, by our choice of \( p \) and \( q \),

\[
w_{p+1} < w_{p+2} < \cdots < w_q < w_p < w_{q+1} < \ldots.
\]

Hence \( v \) covers \( w \) in Bruhat order, and

\[
v_{p+1} < v_{p+2} < \cdots < v_p < v_q < v_{q+1} < \ldots.
\]

Then the only possible \( s > p \) such that \( vt_{ps} \) covers \( v \) is when \( s = q \) (for if \( p < s < q \), then \( vt_{ps} \) has smaller length than \( v \), while if \( s > q \), then the length goes up by more than 1). It follows that the second sum has only one term, namely \( \mathcal{G}_w \). Letting \( T \) be the set of permutations that appear in the first sum then gives the result.